Paper “Derivation of exact spectra of the Schrödinger equation by means of supersymmetry”
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V.E. Adler, Landau Institute for Theoretical Physics

The paper is devoted to applications of the following remarkable transformation. An easy computation proves that if $\psi(x,\lambda)$ is a general solution of the equation

$$\psi'' = (u(x) - \lambda)\psi,$$

and $\psi_0(x)$ is its particular solution corresponding to the value $\lambda = \lambda_0$, then the function

$$\tilde{\psi} = \psi' - w\psi, \quad w = \psi_0'/\psi_0$$

satisfies an equation of the same form with a new potential $\tilde{u} = u - 2w'$. For the first time, this observation was made by Darboux [1], and later on it was rediscovered by Schrödinger [2, 3] in the quantum mechanical context. The sequence of the Darboux transformations is defined by operators $H_n = d^2/dx^2 - u_n$, $Q_n^{\pm} = d/dx \pm w_n$ related by the factorization

$$H_n + \lambda_n = Q_n^+ Q_n^- \quad \mapsto \quad H_{n+1} + \lambda_n = Q_n^- Q_n^+,$$

which is equivalent to the chain of differential-difference equations

$$u_n = w_n' + w_n^2 + \lambda_n, \quad w_n' + w_{n+1}' = w_n^2 - w_{n+1}^2 + \lambda_n - \lambda_{n+1}.$$

Any solution of these equations gives rise to a family of operators H_n with the ψ-functions which can be computed explicitly for all $\lambda = \lambda_n$; the operators $Q_n^{\pm}$ play the role of the creation-annihilation operators. As it turned out, almost all exactly-solvable models of quantum mechanics (harmonic oscillator, Kepler problem, spherical harmonics, reflectionless potentials, Morse, Pöschl–Teller potentials and so on) admit an uniform description within this approach [4, 5]. Moreover,

> *functions w_n corresponding to different n are of the same form and differ only in the values of the parameters.*

The formulation of this *shape-invariance* property is the main result of the Gendenstein’s work. This idea was further developed by Shabat, Veselov, Spiridonov [6, 7, 8] and others, and new families of exactly solvable potentials were introduced (in contrast to the previously known examples, these potentials were defined in terms of the Painlevé transcendents and their generalizations rather than elementary functions).

The Darboux transformation admits generalizations for the non-stationary Scrödinger equation and other spectral problems. It is directly related with the supersymmetry [9, 10, 11] and with the Bäcklund transformations for the nonlinear equations integrable by the inverse scattering method [12, 13, 7]. In 1970–90, the development of these theories was parallel and the Gendenstein’s paper has had a marked impact on these studies.

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