Supplementary Material to the article

"Influence of active loop extrusion on the statistics of triple contacts in the model of interphase chromosomes"

1. Diagram (1). Consider diagram (1) from Fig. 2(1) in the main text, which corresponds to the scenario when there are no loop bases between points i and j. Due to the Markov property of Gaussian chain, vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ are statistically independent, and, taking Eq. (6) from the main text into consideration, we conclude that marginal probability distribution of $\mathbf{R}_2$ is equal to $P_{\text{free}}(\mathbf{R}_2|s_2)$. Therefore, conditional contact probability for diagram (1) is given by

$$p^{(1)}_{jk|ij}(s_1, s_2) = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{s_2^{3/2}}. \tag{S1}$$

As it was indicated in the main text, the angle brackets in Eq. (4) denote averaging over the statistics of random loops. To perform this procedure, it is necessary to know the probability densities of random variables $\{A\}_i$, which parametrize the contributions coming from various classes of diagrams. Since the expression for $p^{(1)}_{jk|ij}(s_1, s_2)$ does not include any loop parameters, in the case of the diagram (1), the corresponding averaging is reduced to the multiplication of the expression (1) by the probability of encountering such a diagram. In short, we need to find the probability that there will not be a single loop base between two randomly selected points of the chain separated by the contour distance $s_1 + s_2$.

To derive the statistical weight of this and all subsequent diagrams, consider a Markov jump process with two states "Loop" and "Gap" , for which time intervals are measured in units of the contour length of our polymer chain, and transitions from one state to another occur with the rates $\alpha_l = \lambda^{-1}$ and $\alpha_g = d^{-1}$. It is clear that the statistics of contour lengths of alternating loops and gaps are equivalent to the statistics of time intervals that such a Markov process spends in the "Loop" and "Gap" states in the course of its stochastic dynamics. Based on this simple analogy, we can express the probability of encountering the first diagram as follows:

$$w^{(1)}(s_1, s_2) = \pi_{gap} Pr[h_1 > s_1 + s_2], \tag{S2}$$

where π_{gap} is the probability to find the statistically stationary Markov process in the state "Gap" at arbitrary point in time, h_1 – the time after which the process first enters the "Loop" state.

Since $\pi_{\text{gap}} = \frac{d}{d+\lambda}$ and $Pr[h > s_1 + s_2] = \int_{s_1+s_2}^{+\infty} dh_1 p_{\text{gap}}(h_1) = e^{-\frac{s_1+s_2}{d}}$, then

$$w^{(1)}(s_1, s_2) = \frac{d}{d+\lambda} e^{-\frac{s_1+s_2}{d}} \approx 1 - \frac{s_1 + s_2 + \lambda}{d}, \tag{S3}$$

where we performed an expansion by the parameters $\lambda/d \ll 1$, $s_1/d \ll 1$ and $s_2/d \ll 1$, keeping only linear corrections to the main contribution.

So, averaging the contribution of diagram (1) to conditional contact probability over statistical weight of this diagram yields

$$\langle p^{(1)}_{jk|ij}(s_1, s_2)\rangle_{\text{loops}} = p^{(1)}_{jk|ij}(s_1, s_2) w^{(1)}(s_1, s_2) = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{s_2^{3/2}} \left(1 - \frac{s_1 + s_2 + \lambda}{d}\right). \tag{S4}$$

2. Diagram (2). In the case of diagram (2) vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ are also statistically independent, so marginal probability distribution of the vector $\mathbf{R}_2$ is equal to $P_{\text{free}}(\mathbf{R}_2|s_2)$. Therefore, conditional contact probability is given by

$$p^{(2)}_{jk|ij}(s_1, s_2, L) = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{s_2^{3/2}}. \tag{S5}$$

Statistical weight of diagram (2) is equal to

$$W^{(2)}(L|s_1, s_2) = \pi_{\text{gap}} Pr[0 < h_1 < s_1 - L] p_{\text{loop}}(L) Pr[h_2 > s_1 + s_2 - L - h_1], \tag{S6}$$

where h_1 is the time after which the Markov process described in the previous section will first enter the "Loop" state provided it was in the "Gap" state at the initial moment, L is the period of time that the process will then spend in the "Loop" state, h_2 is the period of time, which the process will be in the "Gap" state after exiting the "Loop" state.

Since $\pi_{\text{gap}} = \frac{d}{d+\lambda}$, $Pr[0 < h_1 < s_1 - L] = \int_0^{s_1-L} dh_1 p_{\text{gap}}(h_1)$ and $Pr[h_2 > s_1 + s_2 - L - h_1] = \int_{s_1+s_2-L-h_1}^{+\infty} dh_2 p_{\text{gap}}(h_2)$, then in the linear approximation with regards to small dimensionless parameters $\lambda/d \ll 1$, $s_1/d \ll 1$ and $s_2/d \ll 1$ we get

$$W^{(2)}(L|s_1, s_2) \approx \frac{s_1 - L}{d} p_{\text{loop}}(L). \tag{S7}$$

In order to average the contribution of diagram (2) over loop disorder, we should integrate $p^{(2)}_{jk|ij}(s_1, s_2, L)W^{(2)}(L|s_1, s_2)$ over L from 0 to s_1, so

$$\langle p^{(2)}_{jk|ij}(s_1, s_2, L)\rangle_{\text{loops}} = \int_0^{s_1} dL p^{(2)}_{jk|ij}(s_1, s_2, L)W^{(2)}(L|s_1, s_2) = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{s_2^{3/2}} \frac{1}{d} \int_0^{s_1} dL(s_1 - L)p_{\text{loops}}(L). \tag{S8}$$

3. Diagram (3). For the third diagram, vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ as in the previous examples are statistically independent from each other, so the marginal probability distribution of the vector $\mathbf{R}_2$ is equal to $P_{\text{free}}(\mathbf{R}_2|s_2 - L)$. The latter is due to Eq. (3) from the main text and the observation that due to the Markov property of a linear Gaussian chain with free ends, the effect of a loop on this diagram is reduced to a decrease in the effective contour distance between the points j and k. The contribution of the diagram (3) to the conditional contact probability, thus, has the form

$$p^{(3)}_{jk|ij}(s_1, s_2, L) = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{(s_2 - L)^{3/2}}. \tag{S9}$$

By analogy with the previous diagram, the statistical weight of this diagram in a linear approximation by the parameters $\lambda/d \ll 1$, $s_1/d \ll 1$ and $s_2/d \ll 1$ is equal to

$$W^{(3)}(L|s_1, s_2) \approx \frac{s_2 - L}{d} p_{\text{loop}}(L). \tag{S10}$$

Averaging the contribution of the diagram (3) over the length of the loop yields

$$\langle p^{(3)}_{jk|ij}(s_1, s_2, L)\rangle_{\text{loops}} = \int_0^{s_2} dL p^{(3)}_{jk|ij}(s_1, s_2, L)W^{(3)}(L|s_1, s_2) = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{d} \int_0^{s_2} dL \frac{p_{\text{loop}}(L)}{(s_2 - L)^{1/2}}. \tag{S11}$$

4. Diagram (4). Let us move on to a more sophisticated scenario corresponding to the diagram (4). Let $\mathbf{r}_1$ be the vector connecting the point i to the base of the loop, $\mathbf{r}_2$ be the vector connecting the base of the same loop to the point j, and finally $\mathbf{r}_3$ be the vector connecting the base of the loop to the point k. Due to the Markov property of the Gaussian chain, these vectors are statistically independent from each other, and, taking into account Eqs. (2) and (3) from the main text, we can conclude that they have normal statistics with the marginal probability density functions $P_{\text{free}}(\mathbf{r}_1|s_1 - l_1)$, $P_{\text{coil}}(\mathbf{r}_2|l_1, l_1 + l_2)$ and $P_{\text{free}}(\mathbf{r}_3|s_2 - l_2)$, respectively. In this subsection l_1 and l_2 are lengths of segments of the loop which are portrayed at Fig. 2(4) from the main text.

The joint distribution function of the random vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ can be calculated as

$$P^{(4)}_{12}(\mathbf{R}_1, \mathbf{R}_2|l_1, l_2, s_1, s_2) = \langle\delta(\mathbf{R}_1 - \mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{R}_2 - \mathbf{r}_3 + \mathbf{r}_2)\rangle_{\mathbf{r}_1,\mathbf{r}_2} = \tag{S12}$$

$$= \int d^3r_1 \int d^3r_2 \int d^3r_3 \delta(\mathbf{R}_1 - \mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{R}_2 - \mathbf{r}_3 + \mathbf{r}_2)P_{\text{free}}(\mathbf{r}_1|s_1 - l_1)P_{\text{coil}}(\mathbf{r}_2|l_1, l_1 + l_2)P_{\text{free}}(\mathbf{r}_3|s_2 - l_2) = \tag{S13}$$

$$= \int d^3r_2 P_{\text{free}}(\mathbf{R}_1 - \mathbf{r}_2|s_1 - l_1)P_{\text{coil}}(\mathbf{r}_2|l_1, l_1 + l_2)P_{\text{free}}(\mathbf{R}_2 + \mathbf{r}_2|s_2 - l_2) = \tag{S14}$$

$$= \frac{1}{(8\pi^3\sigma^2_{\text{free}}[s_1 - l_1]\sigma^2_{\text{coil}}[l_1, l_1 + l_2]\sigma^2_{\text{free}}[s_2 - l_2])^{3/2}} \times$$

$$\times \int d^3 r_2 \exp\left(-\frac{(\mathbf{R}_1 - \mathbf{r}_2)^2}{2\sigma^2_{\text{free}}[s_1 - l_1]} - \frac{\mathbf{r}_2^2}{2\sigma^2_{\text{coil}}[l_1, l_1 + l_2]} - \frac{(\mathbf{R}_2 + \mathbf{r}_2)^2}{2\sigma^2_{\text{free}}[s_2 - l_2]}\right). \tag{S15}$$

Thus, the partial distribution function of the vector $\mathbf{R}_1$ is equal to

$$P_1^{(4)}(\mathbf{R}_1 | l_1, l_2, s_1, s_2) = \frac{1}{(4\pi^2 \sigma^2_{\text{free}}[s_1 - l_1]\sigma^2_{\text{coil}}[l_1, l_1 + l_2])^{3/2}} \int d^3 r_2 \exp\left(-\frac{(\mathbf{R}_1 - \mathbf{r}_2)^2}{2\sigma^2_{\text{free}}[s_1 - l_1]} - \frac{\mathbf{r}_2^2}{2\sigma^2_{\text{coil}}[l_1, l_1 + l_2]}\right). \tag{S16}$$

From equations (S15) and (S16), and Eq. (5) from the main text, we obtain conditional contact probability for diagram (4):

$$p^{(4)}_{jk|ij}(s_1, s_2, l_1, l_2) = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \left(\frac{s_1(l_1 + l_2) - l_1^2}{s_2 l_1 (s_1 - l_1) + s_1 l_2 (s_2 - l_2)}\right)^{3/2}. \tag{S17}$$

Let us express the lengths of the loop segments as $l_1 = s_1 - h_1$ and $l_2 = L - s_1 + h_1$, where L is the total contour length of the loop, h_1 is the contour distance from the point i to the base of the loop in Fig. 2(4) from the main text. In addition, we denote by h_2 the contour distance from the base of the loop to the nearest loop lying to the right of the point k. In terms of variables L and h_1, the statistical weight of the diagram (4) is equal to

$$W^{(4)}(L, h_1 | s_1, s_2) = \pi_{\text{gap}} p_{\text{loop}}(L) p_{\text{gap}}(h_1) Pr[h_2 > s_1 + s_2 - L - h_1]. \tag{S18}$$

Since $\pi_{\text{gap}} = \frac{d}{\lambda + d}$, $Pr[h_2 > s_1 + s_2 - L - h_1] = \int\limits_{s_1 + s_2 - L - h_1}^{+\infty} dh_2 p_{\text{gap}}(h_2) = e^{-\frac{s_1 + s_2 - L - h_1}{d}}$, then in the linear approximation by the parameters $\lambda/d \ll 1$, $s_1/d \ll 1$ and $s_2/d \ll 1$ we obtain

$$W^{(4)}(L, h_1 | s_1, s_2) \approx \frac{1}{d} p_{\text{loop}}(L). \tag{S19}$$

In order to average the contribution of the diagram (4) over the disorder of loops, it is necessary to integrate the product $p^{(4)}_{jk|ij}(s_1, s_2, s_1 - h_1, L - s_1 + h_1) W^{(4)}(L, h_1 | s_1, s_2)$ over h_1 from $\max(0, s_1 - L)$ to $\min(s_1, s_1 + s_2 - L)$ and over L from 0 to $s_1 + s_2$. This choice of integration limits is dictated by the fact that for the given values of L and h_1 the point j should lie on the loop, while the points i and k should be outside the loop. So

$$\langle p^{(4)}_{jk|ij}(s_1, s_2, s_1 - h_1, L - s_1 + h_1) \rangle_{\text{loops}} =$$

$$= \int\limits_0^{s_1 + s_2} dL \int\limits_{\max(0, s_1 - L)}^{\min(s_1, s_1 + s_2 - L)} dh_1 p^{(4)}_{jk|ij}(s_1, s_2, s_1 - h_1, L - s_1 + h_1) W^{(4)}(L, h_1 | s_1, s_2) = \tag{S20}$$

$$= \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{d} \int\limits_0^{s_1 + s_2} dL p_{\text{loop}}(L) \int\limits_{\max(0, s_1 - L)}^{\min(s_1, s_1 + s_2 - L)} dh_1 \left(\frac{s_1 L - (s_1 - h_1)^2}{s_2(s_1 - h_1) h_1 + s_1 (L - s_1 + h_1)(s_2 - L + s_1 - h_1)}\right)^{3/2}. \tag{S21}$$

5. Diagram (5). Due to the Markov property of the Gaussian chain, the vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ in the case of diagram (5) are statistically independent from each other, and by taking into account Eq. (2) from the main text, we can conclude that the marginal probability density function of the vector $\mathbf{R}_2$ is equal to $P_{\text{free}}(\mathbf{R}_2 | s_2)$. Thus, the conditional contact probability for the diagram (5) has a simple form

$$p^{(5)}_{jk|ij}(s_1, s_2, l_1, l_2) = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{s_2^{3/2}}. \tag{S22}$$

Let us express the lengths of the loop segments indicated in Fig. 2(5) from the main text as $l_1 = qL$, $l_2 = (1 - q)L$, where L is the total contour length of the loop, q is the ratio in which the point i divides the loop. In terms of a pair of variables L and q, the statistical weight of the diagram (5) is equal to

$$W^{(5)}(L, q | s_1, s_2) = \pi_{\text{loop}} \frac{L}{\lambda} p_{\text{loop}}(L) \rho(q) Pr[h_1 > s + 1 + s_2 - (1 - q)L], \tag{S23}$$

where π_{loop} is the probability that the statistically stationary Markov process introduced in section 1 is in the "Loop" state at an arbitrary moment of time (that is, using the language of the original polymer model, this is the probability that the point i lies on a loop), $\frac{L}{\lambda}p_{\text{loop}}(L)$ is the probability density function of the total time that the process will spend in the "Loop" state during the current visit (that is, it is the probability density of the length of the loop on which the point i lies), h_1 – the time after which the process, after leaving the "Loop" state, will return to it again (that is, the contour distance from the base of the loop in the Fig. 2(5) from the main text to the nearest loop lying to the right of the point k), $\rho(q)$ is the distribution function of a random variable q.

Since $\pi_{\text{loop}} = \frac{\lambda}{d+\lambda}$ and $\rho(q) = \theta(q)\theta(1-q)$, where $\theta(q)$ is the Heaviside function, and $Pr[h_1 > s+1+s_2-(1-q)L] = \int_{s+1+s_2-(1-q)L}^{+\infty} dh_1 p_{\text{gap}}(h_1)$, then in the linear approximation by the parameters $\lambda/d \ll 1$, $s_1/d \ll 1$ and $s_2/d \ll 1$ we obtain

$$W^{(5)}(L,q|s_1,s_2) \approx \frac{L}{d}p_{\text{loop}}(L)\theta(q)\theta(1-q). \tag{S24}$$

In order to average the contribution of the diagram (5) over the disorder of loops, it is necessary to integrate the product $p^{(5)}_{jk|ij}(s_1,s_2,qL,(1-q)L)W^{(5)}(L,q|s_1,s_2)$ over q from $\max(0, 1-\frac{s_1}{L})$ up to 1, and over L from 0 to $+\infty$. This choice of integration limits is dictated by the fact that for the given values of L and q, the point i should lie on the loop, while the points j and k should be outside the loop. Therefore

$$\langle p^{(5)}_{jk|ij}(s_1,s_2,l_1,l_2)\rangle_{\text{loops}} = \int_0^{+\infty} dL \int_{\max(0,1-\frac{s_1}{L})}^{1} dq p^{(5)}_{jk|ij}(s_1,s_2,qL,(1-q)L)W^{(5)}(L,q|s_1,s_2) = \tag{S25}$$

$$= \sqrt{\frac{6}{\pi}}\left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{s_2^{3/2}}\frac{1}{d}\left(\int_0^{s_1} dL L p_{\text{loop}}(L) + s_1\int_{s_1}^{+\infty} dL p_{\text{loop}}(L)\right). \tag{S26}$$

6. Diagram (6). For the diagram (6), vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ are statistically independent from each other. Let us denote by $\mathbf{r}_1$ the vector connecting the point j to the base of the loop in Fig. 2(6) from the main text, and by $\mathbf{r}_2$ – the vector connecting the base of the loop to the point k. Due to the Markov property of the Gaussian chain, the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are statistically independent, and taking into account Eqs. (2) and (3) from the main text, we can conclude that they have normal statistics with the marginal probability densities $P_{\text{free}}(\mathbf{r}_1|s_2-l_2)$ and $P_{\text{coil}}(\mathbf{r}_2|l_2,l_1+l_2)$, respectively. In this subsection, l_1 and l_2 are the lengths of the loop segments shown in Fig. 2(6) from the main text. The marginal distribution function of the vector $\mathbf{R}_2$ can be calculated as

$$P_2^{(6)}(\mathbf{R}_2|s_1,s_2,l_1,l_2) = \langle\delta(\mathbf{R}_2-\mathbf{r}_1-\mathbf{r}_2)\rangle_{\mathbf{r}_1,\mathbf{r}_2} = \tag{S27}$$

$$= \int d^3r_1 \int d^3r_2 \delta(\mathbf{R}_2-\mathbf{r}_1-\mathbf{r}_2)P_{\text{free}}(\mathbf{r}_1|s_2-l_2)P_{\text{coil}}(\mathbf{r}_2|l_2,l_1+l_2) = \tag{S28}$$

$$= \int d^3r_1 P_{\text{free}}(\mathbf{r}_1|s_2-l_2)P_{\text{coil}}(\mathbf{R}_2-\mathbf{r}_1|l_2,l_1+l_2) = \tag{S29}$$

$$= \frac{1}{(4\pi D(s_2-l_2+\frac{l_1l_2}{l_1+l_2}))^{3/2}}\exp\left(-\frac{\mathbf{R}_2^2}{4D(s_2-l_2+\frac{l_1l_2}{l_1+l_2})}\right). \tag{S30}$$

Therefore, the conditional contact probability is given by

$$p^{(6)}_{jk|ij}(s_1,s_2,l_1,l_2) = \sqrt{\frac{6}{\pi}}\left(\frac{a}{l_{\text{eff}}}\right)^3\left(\frac{l_1+l_2}{s_2(l_1+l_2)-l_2^2}\right)^{3/2}. \tag{S31}$$

Let us express the lengths of the loop segments as $l_1 = L-s_1-s_2+h_1$ and $l_2 = s_1+s_2-h_1$, where L is the total contour length of the loop, h_1 is the contour distance from the point i to the base of the loop in the Fig. 2(6) from the main text. In terms of a pair of variables L and h_1, the statistical weight of the diagram (6) is

$$W^{(6)}(L,h_1|s_1,s_2) = \pi_{\text{gap}}p_{\text{gap}}(h_1)p_{\text{loop}}(L). \tag{S32}$$

If $\lambda/d \ll 1$, $s_1, s_2 \ll d$, then the following approximate expression can be used in subsequent calculations

$$W^{(6)}(L, h_1|s_1, s_2) \approx \frac{1}{d} p_{\rm loop}(L). \tag{S33}$$

The contribution of the diagram (6) averaged over the disorder of loops is given by

$$\langle p^{(6)}_{jk|ij}(s_1, s_2, l_1, l_2)\rangle_{\rm loops} = \int\limits_{s_1}^{s_1+s_2} dh_1 \int\limits_{s_1+s_2-h_1}^{+\infty} dL p^{(6)}_{jk|ij}(s_1, s_2, L - s_1 - s_2 + h_1, s_1 + s_2 - h_1) W^{(6)}(L, h_1|s_1, s_2) = \tag{S34}$$

$$= \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\rm eff}}\right)^3 \frac{1}{d} \int\limits_{s_1}^{s_1+s_2} dh_1 \int\limits_{s_1+s_2-h_1}^{+\infty} dL p_{\rm loop}(L) \left(\frac{L}{s_2 L - (s_1 + s_2 - h_1)^2}\right)^{3/2}. \tag{S35}$$

The choice of integration limits in the last formula is due to the requirement that for the given values of L and h_1, the point k must lie on the loop, while the points i and j must be outside the loop.

7. Diagram (7). Let $\mathbf{r}_1$ be the vector connecting the point j to the base of the loop in Fig. 2(7) from the main text, and $\mathbf{r}_2$ be the vector connecting the base of the loop to the point k. It is clear that the vector $\mathbf{r}_2$ is statistically independent from the vectors $\mathbf{r}_1$ and $\mathbf{R}_1$, and its probability density has the form $P_{\rm free}(\mathbf{r}_2|s_1+s_2-l_2)$. In this subsection, l_1 and l_2 are the lengths of the loop segments shown in Fig. 2(7) from the main text.

The vectors $\mathbf{r}_1$ and $\mathbf{R}_1$ are not statistically independent from each other, since they connect points belonging to the same loop. Using the analogy between the conformation of an ideal polymer and the trajectory of a random walk, the joint distribution function of these vectors can be expressed as

$$\rho(\mathbf{R}_1, \mathbf{r}_1|l_1, l_2, s_1) = G[\boldsymbol{\mathcal{R}}(s_1) = \mathbf{R}_1, \boldsymbol{\mathcal{R}}(l_2) = \mathbf{R}_1 + \mathbf{r}_1|\boldsymbol{\mathcal{R}}(0) = 0, \boldsymbol{\mathcal{R}}(l_1 + l_2) = 0] = \tag{S36}$$

$$= G[\boldsymbol{\mathcal{R}}(l_2) = \mathbf{R}_1 + \mathbf{r}_1|\boldsymbol{\mathcal{R}}(s_1) = \mathbf{R}_1, \boldsymbol{\mathcal{R}}(0) = 0, \boldsymbol{\mathcal{R}}(l_1 + l_2) = 0] G[\boldsymbol{\mathcal{R}}(s_1) = \mathbf{R}_1|\boldsymbol{\mathcal{R}}(0) = 0, \boldsymbol{\mathcal{R}}(l_1 + l_2) = 0] = \tag{S37}$$

$$= G[\boldsymbol{\mathcal{R}}(l_2) = \mathbf{R}_1 + \mathbf{r}_1|\boldsymbol{\mathcal{R}}(s_1) = \mathbf{R}_1, \boldsymbol{\mathcal{R}}(l_1 + l_2) = 0] G[\boldsymbol{\mathcal{R}}(s_1) = \mathbf{R}_1|\boldsymbol{\mathcal{R}}(0) = 0, \boldsymbol{\mathcal{R}}(l_1 + l_2) = 0], \tag{S38}$$

where $\boldsymbol{\mathcal{R}}(t)$ is the displacement vector of a Brownian particle having diffusivity $D = l_{\rm eff}^2/6$ during the time t, and $G[...|...]$ denotes the probability distribution of the particle displacement at one or more time points, under a given set of conditions. The resulting expression directly follows from the definition of the conditional probability and form the Markov property of Brownian motion.

For the probability distributions entering Eq. (S38) we find

$$G[\boldsymbol{\mathcal{R}}(l_2) = \mathbf{R}_1 + \mathbf{r}_1|\boldsymbol{\mathcal{R}}(s_1) = \mathbf{R}_1, \boldsymbol{\mathcal{R}}(l_1 + l_2) = 0] = \frac{P_{\rm free}(\mathbf{r}_1|l_2 - s_1) P_{\rm free}(-\mathbf{R}_1 - \mathbf{r}_1|l_1)}{P_{\rm free}(-\mathbf{R}_1|l_1 + l_2 - s_1)} = \tag{S39}$$

$$= \left(\frac{l_1 + l_2 - s_1}{4\pi D l_1 (l_2 - s_1)}\right)^{3/2} \exp\left(-\frac{(\mathbf{R}_1 + \mathbf{r}_1)^2}{4Dl_1} - \frac{\mathbf{r}_1^2}{4D(l_2 - s_1)} + \frac{\mathbf{R}_1^2}{4D(l_1 + l_2 - s_1)}\right), \tag{S40}$$

and

$$G[\boldsymbol{\mathcal{R}}(s_1) = \mathbf{R}_1|\boldsymbol{\mathcal{R}}(0) = 0, \boldsymbol{\mathcal{R}}(l_1 + l_2) = 0] = P_{\rm coil}(\mathbf{R}_1|s_1, l_1 + l_2) = \tag{S41}$$

$$= \left(\frac{l_1 + l_2}{4\pi D s_1 (l_1 + l_2 - s_1)}\right)^{3/2} \exp\left(-\frac{(l_1 + l_2)\mathbf{R}_1^2}{4Ds_1(l_1 + l_2 - s_1)}\right). \tag{S42}$$

Substituting Eq. (S40) and Eq. (S42) into Eq. (S38), one obtains

$$\rho(\mathbf{R}_1, \mathbf{r}_1|l_1, l_2, s_1) = \left(\frac{l_1 + l_2}{16\pi^2 D^2 s_1 l_1 (l_2 - s_1)}\right)^{3/2} \exp\left(-\frac{(\mathbf{R}_1 + \mathbf{r}_1)^2}{4Dl_1} - \frac{\mathbf{r}_1^2}{4D(l_2 - s_1)} - \frac{\mathbf{R}_1^2}{4Ds_1}\right). \tag{S43}$$

The joint probability distribution of the random vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ can be calculated as

$$P^{(7)}_{12}(\mathbf{R}_1, \mathbf{R}_2|l_1, l_2, s_1, s_2) = \int d^3 r_1 \int d^3 r_2 \delta(\mathbf{R}_2 - \mathbf{r}_1 - \mathbf{r}_2) P_{\rm free}(\mathbf{r}_2|s_1 + s_2 - l_2) \rho(\mathbf{R}_1, \mathbf{r}_1|l_1, l_2, s_1) = \tag{S44}$$

$$= \int d^3r_2 P_{\mathrm{free}}(\mathbf{r}_2|s_1+s_2-l_2)\rho(\mathbf{R}_1, \mathbf{R}_2-\mathbf{r}_2|l_1,l_2,s_1) = \tag{S45}$$

$$= \left(\frac{l_1+l_2}{64\pi^3 D^3 s_1 l_1 (l_2-s_1)(s_1+s_2-l_2)}\right)^{3/2} \exp\left(-\frac{\mathbf{R}_1^2}{4Ds_1}\right)\cdot \tag{S46}$$

$$\cdot \int d^3r_2 \exp\left(-\frac{\mathbf{r}_2^2}{4D(s_1+s_2-l_2)} - \frac{(\mathbf{R}_1+\mathbf{R}_2-\mathbf{r}_2)^2}{4Dl_1} - \frac{(\mathbf{R}_2-\mathbf{r}_2)^2}{4D(l_2-s_1)}\right). \tag{S47}$$

The marginal probability distribution of the vector $\mathbf{R}_1$ has the form

$$P_1^{(7)}(\mathbf{R}_1|s_1,l_1,l_2) = P_{\mathrm{coil}}(\mathbf{R}_1|s_1,l_1+l_2) = \left(\frac{l_1+l_2}{4\pi D s_1(l_1+l_2-s_1)}\right)^{3/2} \exp\left(-\frac{(l_1+l_2)\mathbf{R}_1^2}{4Ds_1(l_1+l_2-s_1)}\right). \tag{S48}$$

From Eq. (5) from the main text, and Eqs. (S47) and (S48), we find the conditional contact probability for the diagram (7)

$$p_{jk|ij}^{(7)}(s_1,s_2,l_1,l_2) = \sqrt{\frac{6}{\pi}}\left(\frac{a}{l_{\mathrm{eff}}}\right)^3 \left(\frac{l_1+l_2-s_1}{s_2(l_1+l_2)+l_2(2s_1-l_2)-s_1(s_1+s_2)}\right)^{3/2}. \tag{S49}$$

Let us express the lengths of the loop segments as $l_1 = qL$, $l_2 = (1-q)L$, where L is the total contour length of the loop, q is the ratio in which the point i divides the loop. In terms of a pair of variables L and q, the statistical weight of diagram (7) is

$$W^{(7)}(s_1,s_2,L,q) = \pi_{\mathrm{loop}}\frac{L}{\lambda}p_{\mathrm{loop}}(L)\theta(q)\theta(1-q)Pr[h_1 > s_1+s_2-l_2], \tag{S50}$$

where h_1 is the contour distance from the base of the loop in Fig. 2(7) from the main text to the nearest loop lying to the right of the point k. In a linear approximation by the parameters $\lambda/d \ll 1$, $s_1/d \ll 1$ and $s_2/d \ll 1$ we find

$$W^{(7)}(s_1,s_2,L,q) \approx \frac{L}{d}p_{\mathrm{loop}}(L)\theta(q)\theta(1-q). \tag{S51}$$

To average the contribution of the 7-th diagram over the disorder of loops, it is necessary to integrate the product $p_{jk|ij}^{(7)}(s_1,s_2,qL,(1-q)L)W^{(7)}(L,q|s_1,s_2)$ over q from $\max(0, 1-\frac{s_1+s_2}{L})$ up to $1-\frac{s_1}{L}$ and over L from s_1 to $+\infty$. This choice of integration limits is dictated by the fact that for the given values of L and q, the points i and j should lie on the loop, and the point k – outside the loop. So

$$\langle p_{jk|ij}^{(7)}(s_1,s_2,l_1,l_2)\rangle_{\mathrm{loops}} = \int\limits_{s_1}^{+\infty} dL \int\limits_{\max(0,1-\frac{s_1+s_2}{L})}^{1-\frac{s_1}{L}} dq p_{jk|ij}^{(7)}(s_1,s_2,qL,(1-q)L)W^{(7)}(L,q|s_1,s_2) = \tag{S52}$$

$$= \sqrt{\frac{6}{\pi}}\left(\frac{a}{l_{\mathrm{eff}}}\right)^3 \frac{1}{d}\left(\int\limits_{s_1}^{s_1+s_2} dL \int\limits_{0}^{1-\frac{s_1}{L}} dq L p_{\mathrm{loop}}(L)\left(\frac{L-s_1}{s_2L+(1-q)L(2s_1-(1-q)L)-s_1(s_1+s_2)}\right)^{3/2} + \right. \tag{S53}$$

$$\left. + \int\limits_{s_1+s_2}^{+\infty} dL \int\limits_{1-\frac{s_1+s_2}{L}}^{1-\frac{s_1}{L}} dq L p_{\mathrm{loop}}(L)\left(\frac{L-s_1}{s_2L+(1-q)L(2s_1-(1-q)L)-s_1(s_1+s_2)}\right)^{3/2}\right). \tag{S54}$$

8. Diagram (8). Since the diagrams (7) and (8) are equivalent up to the permutation of the points i and k, then

$$P_8(\mathbf{R}_1,\mathbf{R}_2|l_1,l_2,s_1,s_2) = P_7(-\mathbf{R}_2,-\mathbf{R}_1|l_1,l_2,s_2,s_1) = \tag{S55}$$

$$= \left(\frac{l_1+l_2}{64\pi^3 D^3 s_2 l_1(l_2-s_2)(s_1+s_2-l_2)}\right)^{3/2} \exp\left(-\frac{\mathbf{R}_2^2}{4Ds_2}\right)\cdot \tag{S56}$$

$$\int d^3r_1 \exp\left(-\frac{\mathbf{r}_1^2}{4D(s_1+s_2-l_2)} - \frac{(\mathbf{R}_1+\mathbf{R}_2-\mathbf{r}_1)^2}{4Dl_1} - \frac{(\mathbf{R}_1-\mathbf{r}_1)^2}{4D(l_2-s_2)}\right), \tag{S57}$$

where l_1 and l_2 are the lengths of the loop segments shown in Fig. 2(8) from the main text.

Next, comparing diagrams (6) and (8), we see that the partial distribution function of the vector $\mathbf{R}_1$ can be expressed as

$$P_1^{(8)}(\mathbf{R}_1|s_1,s_2,l_1,l_2) = P_1^{(6)}(\mathbf{R}_1|s_2,s_1,l_1+s_2,l_2-s_2) = \tag{S58}$$

$$= \left(\frac{l_1+l_2}{4\pi D(s_1(l_1+l_2)+l_2(2s_2-l_2)-s_2^2)}\right)^{3/2} \exp\left(-\frac{(l_1+l_2)\mathbf{R}_1^2}{4D(s_1(l_1+l_2)+l_2(2s_2-l_2)-s_2^2)}\right). \tag{S59}$$

From Eq. (5) from the main text, and Eqs. (S57) and (S59), we find the conditional contact probability for the diagram (8)

$$p_{jk|ij}^{(8)}(s_1,s_2,l_1,l_2) = \sqrt{\frac{6}{\pi}}\left(\frac{a}{b}\right)^3 \left(\frac{s_1(l_1+l_2)+l_2(2s_2-l_2)-s_2^2}{s_2[s_1(l_1+l_2)+l_2(2s_2-l_2)-s_2(s_1+s_2)]}\right)^{3/2} = \tag{S60}$$

$$= \sqrt{\frac{6}{\pi}}\left(\frac{a}{b}\right)^3 \left(\frac{s_1L+(s_1+s_2-h_1)(s_2-s_1+h_1)-s_2^2}{s_2[s_1L+(s_1+s_2-h_1)(s_2-s_1+h_1)-s_2(s_1+s_2)]}\right)^{3/2}. \tag{S61}$$

Let us express the lengths of the loop segments as $l_2 = s_1+s_2-h_1$ and $l_1 = L-s_1-s_2+h_1$, where L is the total contour length of the loop in Fig. 2(8) from the main text, h_1 is the contour distance from the point i to the base of this loop. In terms of a pair of variables L and h_1, the statistical weight of the diagram (8) is

$$W^{(8)}(L,h_1|s_1,s_2) = \pi_{\text{gap}} p_{\text{gap}}(h_1) p_{\text{loop}}(L). \tag{S62}$$

If $\lambda/d \ll 1$, $s_1, s_2 \ll d$, then an approximate expression can be used in subsequent calculations

$$W^{(8)}(L,h_1|s_1,s_2) \approx \frac{1}{d} p_{\text{loop}}(L). \tag{S63}$$

To average the contribution of the diagram (8) over the disorder of loops, it is necessary to integrate the product $p_{jk|ij}^{(8)}(s_1,s_2,s_1+s_2-h_1,L-s_1-s_2+h_1)W^{(8)}(L,h_1|s_1,s_2)$ over L from $s_1+s_2-h_1$ to $+\infty$ and over h_1 from 0 to s_1. This choice of integration limits is dictated by the fact that for the given values of L and h_1 the points j and k should lie on the loop, and the point i should lie outside the loop. So

$$\langle p_{jk|ij}^{(8)}(s_1,s_2,l_1,l_2)\rangle_{\text{loops}} = \int_0^{s_1} dh_1 \int_{s_1+s_2-h_1}^{+\infty} dL p_{jk|ij}^{(8)}(s_1,s_2,s_1+s_2-h_1,L-s_1-s_2+h_1)W^{(8)}(L,h_1|s_1,s_2) = \tag{S64}$$

$$= \sqrt{\frac{6}{\pi}}\left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{d}\int_0^{s_1} dh_1 \int_{s_1+s_2-h_1}^{+\infty} dL p_{\text{loop}}(L) \left(\frac{s_1L+(s_1+s_2-h_1)(s_2-s_1+h_1)-s_2^2}{s_2[s_1L+(s_1+s_2-h_1)(s_2-s_1+h_1)-s_2(s_1+s_2)]}\right)^{3/2}. \tag{S65}$$

9. Diagram (9). In the case of the diagram (9) (see Fig. 2(9) from the main text) the random vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ are not statistically independent, since they connect the points lying on the same loop. The joint probability distributions of these vectors can be represented as

$$P_{12}^{(9)}(\mathbf{R}_1,\mathbf{R}_2|L,s_1,s_2) = G[\boldsymbol{\mathcal{R}}(s_1)=\mathbf{R}_1, \boldsymbol{\mathcal{R}}(s_1+s_2)=\mathbf{R}_1+\mathbf{R}_2|\boldsymbol{\mathcal{R}}(0)=\mathbf{0}, \boldsymbol{\mathcal{R}}(L)=\mathbf{0}] = \tag{S66}$$

$$= G[\boldsymbol{\mathcal{R}}(s_1+s_2)=\mathbf{R}_1+\mathbf{R}_2|\boldsymbol{\mathcal{R}}(s_1)=\mathbf{R}_1, \boldsymbol{\mathcal{R}}(0)=\mathbf{0}, \boldsymbol{\mathcal{R}}(L)=\mathbf{0}]G[\boldsymbol{\mathcal{R}}(s_1)=\mathbf{R}_1|\boldsymbol{\mathcal{R}}(0)=\mathbf{0}, \boldsymbol{\mathcal{R}}(L)=\mathbf{0}] = \tag{S67}$$

$$= G[\boldsymbol{\mathcal{R}}(s_1+s_2)=\mathbf{R}_1+\mathbf{R}_2|\boldsymbol{\mathcal{R}}(s_1)=\mathbf{R}_1, \boldsymbol{\mathcal{R}}(L)=\mathbf{0}]G[\boldsymbol{\mathcal{R}}(s_1)=\mathbf{R}_1|\boldsymbol{\mathcal{R}}(0)=\mathbf{0}, \boldsymbol{\mathcal{R}}(L)=\mathbf{0}], \tag{S68}$$

where L is the length of the loop, $\boldsymbol{\mathcal{R}}(t)$ is the displacement vector of a Brownian particle with a diffusion coefficient $D = l_{\text{eff}}^2/6$ for the time t, and $G[...|...]$ denotes the probability distribution of the particle displacements at one or more time points, under a given set of conditions.

For the probability distributions entering Eq. (S68), we find

$$G[\boldsymbol{\mathcal{R}}(s_1+s_2)=\mathbf{R}_1+\mathbf{R}_2|\boldsymbol{\mathcal{R}}(s_1)=\mathbf{R}_1,\boldsymbol{\mathcal{R}}(L)=\mathbf{0}]=\frac{P_{\mathrm{free}}(\mathbf{R}_2|s_2)P_{\mathrm{free}}(-\mathbf{R}_1-\mathbf{R}_2|L-s_1-s_2)}{P_{\mathrm{free}}(-\mathbf{R}_1|L-s_1)}= \tag{S69}$$

$$=\left(\frac{L-s_1}{4\pi D s_2(L-s_1-s_2)}\right)^{3/2}\exp\left(-\frac{(\mathbf{R}_1+\mathbf{R}_2)^2}{4D(L-s_1-s_2)}-\frac{\mathbf{R}_2^2}{4Ds_2}+\frac{\mathbf{R}_1^2}{4D(L-s_1)}\right), \tag{S70}$$

and

$$G[\boldsymbol{\mathcal{R}}(s_1)=\mathbf{R}_1|\boldsymbol{\mathcal{R}}(0)=0,\boldsymbol{\mathcal{R}}(L)=0]=P_{\mathrm{coil}}(\mathbf{R}_1|s_1,L)=\left(\frac{L}{4\pi D s_1(L-s_1)}\right)^{3/2}\exp\left(-\frac{L\mathbf{R}_1^2}{4Ds_1(L-s_1)}\right). \tag{S71}$$

Next, substituting (S70) and (S71) into (S68), one obtains

$$P_{12}^{(9)}(\mathbf{R}_1,\mathbf{R}_2|L,s_1,s_2)=\left(\frac{L}{16\pi^2D^2s_1s_2(L-s_1-s_2)}\right)^{3/2}\exp\left(-\frac{(\mathbf{R}_1+\mathbf{R}_2)^2}{4D(L-s_1-s_2)}-\frac{\mathbf{R}_2^2}{4Ds_2}-\frac{\mathbf{R}_1^2}{4Ds_1}\right), \tag{S72}$$

and

$$P_1^{(9)}(\mathbf{R}_1|L,s_1)=P_{\mathrm{coil}}(\mathbf{R}_1|s_1,L)=\left(\frac{L}{4\pi D s_1(L-s_1)}\right)^{3/2}\exp\left(-\frac{\mathbf{R}_1^2}{4D(L-s_1)}\right). \tag{S73}$$

From Eqs. (S72) and (S73) and Eq. (5) from the main text, we obtain the conditional contact probability for the diagram (9)

$$p_{jk|ij}^{(9)}(s_1,s_2,L)=\sqrt{\frac{6}{\pi}}\left(\frac{a}{l_{\mathrm{eff}}}\right)^3\left(\frac{L-s_1}{s_2(L-s_1-s_2)}\right)^{3/2}. \tag{S74}$$

Statistical weight of the diagram (9) is equal to

$$W^{(9)}(L,q|s_1,s_2)=\pi_{\mathrm{loop}}\frac{L}{\lambda}p_{\mathrm{loop}}(L)\theta(q)\theta(1-q), \tag{S75}$$

where q is the ratio in which the point i divides the loop. In the linear approximation by the parameter $\lambda/d\ll 1$ we find

$$W^{(9)}(L,q|s_1,s_2)\approx\frac{L}{d}p_{\mathrm{loop}}(L)\theta(q)\theta(1-q). \tag{S76}$$

To average the contribution of the diagram (9) over the disorder of loops, it is necessary to integrate the product $p_{jk|ij}^{(9)}(s_1,s_2,L)W^{(9)}(L,q|s_1,s_2)$ over q from 0 up to $1-\frac{s_1+s_2}{L}$ and over L from s_1+s_2 to $+\infty$. This choice of integration limits is dictated by the fact that for the given values of L and q, all three points i and j and k must lie on the loop. So

$$\langle p_{jk|ij}^{(9)}(s_1,s_2,L)\rangle_{\mathrm{loops}}=\int\limits_{s_1+s_2}^{+\infty}dL\int\limits_0^{1-\frac{s_1+s_2}{L}}dq\,p_{jk|ij}^{(9)}(s_1,s_2,L)W^{(9)}(L,q|s_1,s_2)= \tag{S77}$$

$$=\sqrt{\frac{6}{\pi}}\left(\frac{a}{l_{\mathrm{eff}}}\right)^3\frac{1}{d}\int\limits_{s_1+s_2}^{+\infty}dL\left(\frac{L-s_1}{s_2}\right)^{3/2}\frac{p_{\mathrm{loop}}(L)}{\sqrt{L-s_1-s_2}}. \tag{S78}$$

10. Conditional contact probability. Contributions of the diagrams given by Eqs. (S4), (S8), (S11), (S21), (S26), (S35), (S54), (S65) and (S78), after being substituted into Eq. (4) from the main text, yield Eq. (7) from the main text, where

$$F(z_1,z_2)=-z_1-z_2-1+z_1^2\int\limits_0^1 d\tilde{L}(1-\tilde{L})\lambda p_{\mathrm{loop}}(\lambda z_1\tilde{L})+z_2^2\int\limits_0^1 d\tilde{L}\frac{\lambda p_{\mathrm{loop}}(\lambda z_2\tilde{L})}{(1-\tilde{L})^{1/2}}+ \tag{S79}$$

$$+ z_2^2 \int\limits_0^{1+\frac{z_1}{z_2}} d\tilde{L} \int\limits_{\max(0;\frac{z_1}{z_2}-\tilde{L})}^{\min(\frac{z_1}{z_2};\frac{z_1}{z_2}+1-\tilde{L})} d\tilde{h}\lambda p_{\text{loop}}(\lambda z_2 \tilde{L}) \left[\frac{\frac{z_1}{z_2}\tilde{L} - (\frac{z_1}{z_2}-\tilde{h})^2}{(\frac{z_1}{z_2}-\tilde{h})\tilde{h} + \frac{z_1}{z_2}(\tilde{L}-\frac{z_1}{z_2}+\tilde{h})(1-\tilde{L}+\frac{z_1}{z_2}-\tilde{h})} \right]^{3/2} + \tag{S80}$$

$$+ z_1^2 \int\limits_0^1 d\tilde{L}\tilde{L}\lambda p_{\text{loop}}(\lambda z_1 \tilde{L}) + z_1^2 \int\limits_1^{+\infty} d\tilde{L}\lambda p_{\text{loop}}(\lambda z_1 \tilde{L}) + z_2^2 \int\limits_{\frac{z_1}{z_2}}^{1+\frac{z_1}{z_2}} d\tilde{h} \int\limits_{1+\frac{z_1}{z_2}-\tilde{h}}^{+\infty} d\tilde{L}\lambda p_{\text{loop}}(\lambda z_2 \tilde{L}) \left[\frac{\tilde{L}}{\tilde{L} - (1+\frac{z_1}{z_2}-\tilde{h})^2} \right]^{3/2} + \tag{S81}$$

$$+ \left(\frac{z_2}{z_1}\right)^{3/2} z_1^2 \int\limits_1^{1+\frac{z_2}{z_1}} d\tilde{L} \int\limits_0^{1-\frac{1}{\tilde{L}}} dq\tilde{L}\lambda p_{\text{loop}}(\lambda z_1 \tilde{L}) \left[\frac{\tilde{L}-1}{\frac{z_2}{z_1}\tilde{L} + (1-q)\tilde{L}(2-(1-q)\tilde{L}) + 1 + \frac{z_2}{z_1}} \right]^{3/2} + \tag{S82}$$

$$+ \left(\frac{z_2}{z_1}\right)^{3/2} z_1^2 \int\limits_{1+\frac{z_2}{z_1}}^{+\infty} d\tilde{L} \int\limits_{1-\frac{1}{\tilde{L}}(1+\frac{z_2}{z_1})}^{1-\frac{1}{\tilde{L}}} dq\tilde{L}\lambda p_{\text{loop}}(\lambda z_1 \tilde{L}) \left[\frac{\tilde{L}-1}{\frac{z_2}{z_1}\tilde{L} + (1-q)\tilde{L}(2-(1-q)\tilde{L}) + 1 + \frac{z_2}{z_1}} \right]^{3/2} + \tag{S83}$$

$$+ z_1^2 \int\limits_0^1 d\tilde{h} \int\limits_{1+\frac{z_2}{z_1}-\tilde{h}}^{+\infty} d\tilde{L}\lambda p_{\text{loop}}(\lambda z_1 \tilde{L}) \left[\frac{\tilde{L} + (1+\frac{z_2}{z_1}-\tilde{h})(\frac{z_2}{z_1}-1+\tilde{h}) - \left(\frac{z_2}{z_1}\right)^2}{\tilde{L} + (1+\frac{z_2}{z_1}-\tilde{h})(\frac{z_2}{z_1}-1+\tilde{h}) - \frac{z_2}{z_1}(1+\frac{z_2}{z_1})} \right]^{3/2} + \tag{S84}$$

$$+ z_2^2 \int\limits_{1+\frac{z_1}{z_2}}^{+\infty} d\tilde{L} \frac{\lambda p_{\text{loop}}(\lambda z_2 \tilde{L})}{\sqrt{\tilde{L}-\frac{z_1}{z_2}-1}} \left(\tilde{L} - \frac{z_1}{z_2}\right)^{3/2}. \tag{S85}$$

11. Marginal contact probability. The marginal contact probability between two pints j and k is given by the following expression (see [39] in the main text)

$$p_{jk}(s_2) = Pr[R_2 < a] = \sqrt{\frac{6}{\pi}} \left(\frac{a}{l_{\text{eff}}}\right)^3 \frac{1}{s_2^{3/2}} \left(1 + \frac{\lambda}{d} f\left(\frac{s_2}{\lambda}\right)\right), \tag{S86}$$

where

$$f(z_2) = -1 - z_2 + z_2^2 \int\limits_0^1 d\tilde{L} \frac{1+2\tilde{L}}{(1-\tilde{L})^{1/2}} \lambda p_{\text{loops}}(\lambda z_2 \tilde{L}) + z_2^2 \int\limits_1^{+\infty} d\tilde{L} \frac{\tilde{L}^{1/2}(\tilde{L}+2)}{(\tilde{L}-1)^{1/2}} \lambda p_{\text{loops}}(\lambda z_2 \tilde{L}). \tag{S87}$$