

## DILUTED GENERALIZED RANDOM ENERGY MODEL

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We introduce a layered random spin model, equivalent to the Generalized Random Energy Model (GREM). In analogy with diluted spin systems, a diluted GREM (DGREM) is introduced. It can be applied to calculate approximately thermodynamic properties of spin glass models in low dimensions. For Edwards -Anderson model it gives correct critical dimension and 5% accuracy for ground state energy in 2d.

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Derrida's Random Energy Model (REM) [1] was introduced as an archetype spin glass [2] model. In recent years it is becoming more and more popular. It has been applied in many fields of physics, biology and even in information theory [3, 4]. The generalization of the REM (called Generalized Random Energy Model, GREM) was introduced in ref. [5]. It was used to solve approximately other spin glass systems [6,7]. Unfortunately, the accuracy to describe other spin glass systems was not much better than for the REM. In this work we introduce a diluted spin model which thermodynamically resembles GREM (in the case of large coordination number it is exactly equivalent to GREM), then construct some new model of energy configurations – DGREM. In some (practically important) cases our spin model is thermodynamically exactly equivalent to DGREM.

Even the simpler diluted REM (DREM) [8, 9] has proven to be a good approximation for models in low dimensions ( $d = 1, 2, 3$ ). This important fact was observed in [10], where by information-theoretical arguments (mathematically leading to a DREM) a percolation threshold was found.

In the DREM one has  $N$  Ising spins interacting with each other in the  $z$  (randomly chosen from all the possible  $C_N^p = N!/p!(N-p)!$ )  $p$ -plets of Ising spins and quenched random couplings  $\tau_{i_1, \dots, i_p}$  having values  $\pm 1$ .

The Hamiltonian reads

$$\mathcal{H} = - \sum_{(1 \leq i_1, \dots, i_p \leq N)=1}^z \tau_{i_1, \dots, i_p} \sigma_{i_1} \cdots \sigma_{i_p}. \quad (1)$$

At high temperatures the system is in the para magnetic phase and

$$\frac{F}{N} = -dT \ln \cosh \beta - T \ln 2, \quad (2)$$

where  $\beta = 1/T$ . Below the critical temperature  $T_c = 1/\beta_c$  the system freezes in a spin-glass phase with internal energy  $U/N = -d \tanh \beta_c$  and vanishing entropy  $S = 0$ . Here  $\tanh \beta_c = f(d)$  involves a function  $f(x)$  defined by the implicit equation

$$\frac{1}{2}(1+f) \ln(1+f) + \frac{1}{2}(1-f) \ln(1-f) = \frac{\ln 2}{x}. \quad (3)$$

For the ground state energy of the Edwards - Anderson (EA) model on a hyper cubic lattice in  $d$ -dimensions ( $z = Nd$ )

$$-\frac{E}{N} = f(d)d. \quad (4)$$

In 2d Esq. (4) gives  $E \approx -1.5599$ , which is close to result of Monte-Carlo simulation for the case of random  $\pm 1$  couplings [11]  $E/N = -1.4015 \pm 0.0008$ . This estimate by formula (3) was done by Derrida in his original work [1], long before the introduction of the DREM in reference [8].

Let us now construct some spin model, which has properties like the GREM. It is very important to have spin representation for GREM (for example - to construct temporal dynamics).

We consider a stacked system consisting of  $M$  planes with spin  $\sigma_i^k$  ordered along a "vertical" axis. In plane (layer)  $k$  there are  $N_k$  spins. So spins in the layer  $1 < k < M$  interact with spins from the layers  $k \pm 1$ , the first layer interacts with the spins of second layer, spins from the layer  $M$  interact with each other. We have the Hamiltonian

$$\begin{aligned} \mathcal{H} = & - \sum_{(1 \leq i_1, \dots, i_p \leq N_M)}^{z_M} \tau_{i_1, \dots, i_p} \sigma_{i_1}^M \dots \sigma_{i_p}^M - \\ & - \sum_{k=1}^{M-1} \sum_{(1 \leq i_1, \dots, i_{p/2} \leq N_{k-1}, 1 \leq j_1, \dots, j_{p/2} \leq N_k)}^{z_k} \tau_{i_1, \dots, i_p} \sigma_{i_1}^{k-1} \dots \sigma_{i_{p/2}}^{k-1} \sigma_{j_1}^k \dots \sigma_{j_{p/2}}^k. \end{aligned} \quad (5)$$

Let us now introduce some (equivalent) GREM like model. We consider some  $M$  level hierarchic tree. At first level there are  $2_1^N$  branches. At second level every old branch fractures to  $2^{N_2}$  new ones and so on. At the level  $M$  there are  $2^N$  branches, where  $N = \sum_{i=1, M}^M N_i$  Energy configurations of our system are located on the ends of  $M$ -th level branches. On every branch of level  $i$  there are located  $2^{N_i}$  random variables  $\epsilon_i^\alpha$  with the distribution

$$\rho_0(\epsilon_i^\alpha, z_i) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dk \exp[-k\epsilon_i^\alpha + z_i \ln \cosh k]. \quad (6)$$

This is a distribution for a sum of  $z_k$  random  $\pm 1$  variables. So  $z_k$  resembles number of couplings in our diluted spin models. With any energy configuration are connected  $M$  branches. We define configuration energy as a sum (along the path on the tree, connected with chosen energy configuration) of these  $M$  variables  $\epsilon_i^\alpha$ . We see a usual picture of GREM, where random variable are distributed according to (6) instead of normal partition.

We can consider the case of large  $M$  with smooth distribution of  $z_k$  and  $N_k$ . In this case we can introduce continues variable  $v = k/M$  between 0 and 1, labeling the level of planes and define distributions

$$z_k \equiv dz = z dv, \quad N_k \equiv dN = n'(v)dv \quad dv = \frac{1}{M}, \quad (7)$$

where  $n(v)$  is a given function (entropy in bits). The variable  $v$  ( $0 < v < 1$ ) parameterizes the level of the hierarchical tree and  $z$  is a parameter (for our spin system  $z$  is a total number of couplings and parameter  $v$  labeling the level of planes).

Of course, our function  $n(v)$  should be monotonic. Total number of energy configurations is  $2^{n(0)}$  and  $n(0) = N$ . We have, that  $2^N$  energy levels  $E$  of our hierarchic model are distributed by partition  $\rho(E) = \rho_0(E, z)$ . If two configurations (in our GREM like model) meet at level of hierarchy  $v$ , they have  $zv$  common random variables. The energy difference between two configurations is related to  $z(1 - v)$  non-common random variables. Therefore the distribution function of two energies  $E_1, E_2$  reads

$$\rho_2(E_1 - E_2) = \rho_0((E_1 - E_2), 2z(1 - v)) \exp(\ln 2n(v)). \quad (8)$$

At high temperatures our system is in the para magnetic phase. The free energy is given by eq. (2). When we decrease the temperature, two situations are possible: first,  $dz/dN \equiv z/n'(v)$  decreases monotonically with  $v$ ; second, it has a local maxim.

In the first case the system has no sharp phase transition but it freezes gradually. At temperature  $T = 1/\beta$  all level with  $0 \leq v \leq v_f(T)$  are frozen; they are in the spin glass phase. The levels with  $v_f < v \leq 1$  are in the paramagnetic phase;  $v_f$  is defined as the solution  $v_f = v$  of equation

$$\tanh \beta = f\left(\frac{z}{n'(v)}\right). \quad (9)$$

Using this relationship between  $\beta$  and  $v$  we can later use functions  $v(\beta)$  and  $\beta(v)$ . For every finite  $\beta$  the value of  $v(\beta)$  lies between zero and unity. When  $T \rightarrow \infty$ ,  $v(\beta) \rightarrow 0$  and when  $T \rightarrow 0$ ,  $v(\beta) \rightarrow v_0 > 0$ . So even in this limit some part of spins stay in their para magnetic phase. Let us point out that this partial freezing only is possible in the Diluted GREM, and not in the original GREM. For the free energy we obtain ( there is no factor  $N$  in it):

$$-\beta F = z(1 - v(\beta)) \ln \cosh \beta + n(v(\beta)) \ln 2 + z\beta \int_0^{v(\beta)} dv_1 f\left(\frac{z}{n'(v_1)}\right). \quad (10)$$

The first two terms in the right hand side describes the para magnetic fraction of free energy ( $n(v(\beta)) \ln 2$  just is the entropy), while the last one describe the fraction of spins frozen in a glassy configuration (it resembles equation (3) with  $z/f'(v_1)$  instead of  $d$ ). In the second case (function  $n'(v)$  is not monotonic) the system has a sharp first order phase transition at a finite temperature  $T_2$ . Below  $T_2$  freezing occurs drastically for all levels  $v < v_2$ , where  $v_2 \equiv v(\beta_2)$  is defined by the equation  $n'(v_2) = N - n(v_2)$ . We used the fact that  $n(0) = N$ . The transition temperature  $T_2 = 1/\beta_2$  follows from  $\tanh(\beta_2) = f(z/n'(v_2))$ . For temperatures  $T < T_2$  the free energy reads

$$\begin{aligned} -\frac{\beta F}{N} &= z(1 - v(\beta)) \ln \cosh \beta + n(v(\beta)) \ln 2 + \\ &+ zv(\beta_2)\beta f\left(\frac{z}{n'(v_2)}\right) + z\beta \int_{v_2}^{v(\beta)} dv_1 f\left(\frac{z}{n'(v_1)}\right). \end{aligned} \quad (11)$$

To construct the spin Hamiltonian by means of chain of subsystems for this case is still an open problem. Let us now consider a possible approximation to the Edwards-Anderson model, following the idea's presented in ref. [6].  $d$ -dimensional case our  $2^N$  energy levels  $E$  are distributed according to the law

$$\rho(\epsilon) = \rho_0(\epsilon, Nd) \quad (12)$$

with  $\rho_0$  defined in eq. (6). Comparing with (6) one immediately notices that this is exactly equivalent to a DGREM with the choice  $E = \epsilon$ ,  $z = Nd$ . Let us now consider the distribution of  $\epsilon_1 - \epsilon_2$ . Following the arguments presented in reference [6], we find that

$$z = Nd, \quad n(v) = \frac{Ns(-vdN)}{\ln 2}. \quad (13)$$

We see, that variable  $v$  corresponds to energy per bond in ferromagnetic model. We can remember from definition of temperature  $ds/dE = 1/\tau \equiv \beta$ . At given  $\tilde{\beta}_1$  we can define corresponding  $v_1$  as minus energy per bond for ferromagnetic model at temperature  $1/\tilde{\beta}_1$ :

$$v_1 = -\frac{E(\tilde{\beta}_1)}{Nd}. \quad (14)$$

We obtain for the free energy

$$-\frac{\beta F}{Nd} = (1 - v(\beta)) \ln \cosh \beta + s(v(\beta)) + \beta \int_0^{v(\beta)} dv_1 f\left(\frac{\ln 2}{\beta}\right). \quad (15)$$

Integrating by parts in the last term we get

$$-\frac{\beta F}{Nd} = (1 - v(\beta)) \ln \cosh \beta + s(v(\beta)) - \beta \int_0^{\tilde{\beta}} d\tilde{\beta}_1 \frac{2v_1(\tilde{\beta}_1)}{\ln \frac{1+y}{1-y}} + v(\beta)\beta y(\tilde{\beta}), \quad (16)$$

where  $y$  as a function of  $\tilde{\beta}_1$  is defined from the equation

$$y = f\left(\frac{\ln 2}{\tilde{\beta}_1}\right) \quad (17)$$

function  $v(\beta)$  is defined from (9),(13), and  $v_1(\tilde{\beta}_1)$  is minus energy per bond in ferromagnetic model at temperature  $\tilde{\beta}_1$ . In the equation (16) value of the  $\tilde{\beta}$  is connected with the given  $\beta$  via the equation  $\tanh(\beta) = f(\ln 2/\beta)$ . In the limit of zero temperature this reduces to

$$-\frac{\beta F}{N} = d\left[1 - \frac{2}{d} \int_0^{\ln 2} \frac{d\tilde{\beta}_1 E(\tilde{\beta}_1)}{\ln 1 + y(\tilde{\beta}_1)/1 - y(\tilde{\beta}_1)}\right] \quad (18)$$

He  $E(\tilde{\beta}) = |U|$  is minus the energy in the ferromagnetic model,  $y(\tilde{\beta})$  is defined from (17) and function  $f(x)$  from the (3). A calculation of the ground state energy for 2d EA model using (18) gives  $E = -1.4763$ . For the case of other models one can use numerical data for the ferromagnetic system.

This simple approximation to the ground state energy of disordered systems should be efficient at low dimensions.

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