

BRANE-BLACK HOLE CORRESPONDENCE AND ASYMPTOTICS OF QUANTUM SPECTRUM

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We discuss the asymptotic properties of quantum states density for fundamental (super) membrane in the semiclassical approach. The matching of BPS part of spectrum for superstring and supermembrane gives the possibility to get stringy results via membrane calculations and vice versa. The brane-black hole correspondence (on the level of black hole states and brane microstates) is also studied.

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1. It has been realized recently that there are very deep connections between fundamental (super) membrane and (super) string theory. In particular, it has been shown that the BPS spectrum of states for type IIB string on a circle is in correspondence with the BPS spectrum of fundamental compactified supermembrane [1, 2]. Remarkable progress has been made towards establishing the string-black hole correspondence relevant for the extreme black hole [3]. The entropy associated with the BPS states is then identical to the Bekenstein-Hawking entropy defined by the horizon area. The idea of string-black hole correspondence has been formulated also as the correspondence principle in Ref. [4].

2. That is the purpose of this work to look to some of these questions from fundamental supermembrane point of view. We start from the semiclassical free energy for fundamental compactified supermembranes (which is known to be divergent) embedded in flat D -dimensional manifolds with topologies $\mathcal{M} = S^1 \otimes T^d \otimes R^{D-d-1}$ (T^d is the d -dimensional torus). First of all we remind that for simplest quantum field model the free energy has the form [5, 6]

$$\mathcal{F}^{(b,f)}(\beta) = -\pi^d (\det A)^{1/2} \int_0^\infty ds (2s)^{-(D-d+2)/2} \Xi^{(b,f)}(s, \beta) \times \\ \times \Theta \left[\begin{matrix} g \\ 0 \end{matrix} \right] (0|\Omega) \exp \left(-\frac{sM^2}{2\pi} \right), \quad (1)$$

where

$$\Xi^{(b)}(s, \beta) = \theta_3 \left(0 \left| \frac{i\beta^2}{2s} \right. \right) - 1, \quad \Xi^{(f)}(s, \beta) = 1 - \theta_4 \left(0 \left| \frac{i\beta^2}{2s} \right. \right), \quad (2)$$

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and $\theta_3(\nu|\tau)$ and $\theta_4(\nu|\tau) = \theta_3(\nu + \frac{1}{2}|\tau)$ are the Jacobi theta functions. Here $\mathcal{A} = \text{diag}(R_1^{-2}, \dots, R_d^{-2})$ is a $d \times d$ matrix, the global parameters R_j characterizing the non-trivial topology of \mathcal{M} appear in the theory owing to the fact that coordinates $x_j (j = 1, \dots, d)$ obey the conditions $0 \leq x_j < 2\pi R_j$. The number of topological configurations of quantum fields is equal to the number of elements in group $H^1(\mathcal{M}; \mathbb{Z}_2)$, first cohomology group with coefficients in \mathbb{Z}_2 . The multiplet $g = (g_1, \dots, g_d)$ defines the topological type of field (i.e., the corresponding twist), and depending on the field type chosen in \mathcal{M} , $g_j = 0$ or $1/2$. In our case $H^1(\mathcal{M}; \mathbb{Z}_2) = \mathbb{Z}_2^d$ and so the number of topological configurations of real scalars (spinors) is 2^d . We follow the notations and treatment of Ref. [7] and introduce the theta function with characteristics a, b for $a, b \in \mathbb{Z}^d$,

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\Omega) = \sum_{\mathbf{n} \in \mathbb{Z}^d} \exp[i\pi(\mathbf{n} + \mathbf{a})\Omega(\mathbf{n} + \mathbf{a}) + i2\pi(\mathbf{n} + \mathbf{a})(z + \mathbf{b})], \quad (3)$$

in this connection $\Omega = (is/2\pi^2)\text{diag}(R_1^2, \dots, R_d^2)$. The above method of the free energy calculation admits the subsequent development for extended objects. We shall assume that free energy is equivalent to a sum of the free energies of quantum fields which present in the modes of a membrane. The factor $\exp(-sM^2/2\pi)$ in Eq. (1) should be understood as $\text{Tr} \exp(-sM^2/2\pi)$, where M is the mass operator of membrane and the trace is over infinite set of Bose-Fermi oscillators $N_{\mathbf{n}}^{(b)}, N_{\mathbf{n}}^{(f)}$.

3. For the noncompactified supermembrane the question of reliability of the semiclassical approximation is not absolutely clear [8]. The discrete part of the supermembrane spectrum propagating in eleven-dimensional Minkowski space-time can be written in the form (see for detail Refs. [8, 5])

$$M^2 = \sum_{j=1}^8 \sum_{\mathbf{n} \in \mathbb{Z}^2 / \{0\}} \omega_{\mathbf{n}} (N_{\mathbf{n}_j}^{(b)} + N_{\mathbf{n}_j}^{(f)}), \quad (4)$$

where

$$\omega_{\mathbf{n}} = \sqrt{(n_1\pi/a)^2 + (n_2\pi/b)^2}, \quad (5)$$

and $a = \pi R_1, b = \pi R_2$. Thus as a result we have

$$\text{Tr} \sum_{\mathbf{n} \in \mathbb{Z}^4 / \{0\}} \exp\left(-\frac{s}{2\pi} M^2\right) = [H_+(\Omega)H_-(\Omega)]^8, \quad (6)$$

where

$$H_{\pm}(\Omega) = \prod_{\mathbf{n} \in \mathbb{Z}^4 / \{0\}} \left\{ 1 \pm \exp[-(\mathbf{n}, \Omega\mathbf{n})^{1/2}] \right\}^{(\pm 1)}, \quad (7)$$

and $\Omega = (s^2/4)\text{diag}(a^{-2}, b^{-2})$.

For generating functions $H_{\pm}(\Omega)$, $\Omega = z\text{diag}(1, \dots, 1)$, ($z = t + 2\pi x$) in the half-plane $\Re z > 0$ there exists an asymptotic expansion uniformly in x as $t \rightarrow 0$, provided $|\arg z| \leq \frac{\pi}{4}$ and $|x| \leq \frac{1}{2}$ and given by [9]

$$H_+(\Omega) = \exp \left\{ [A\Gamma(p)\zeta_-(1+p)z^{-p} - Z_p(0)\log 2 + O(t^{c+})] \right\}, \quad (8)$$

$$H_-(\Omega) = \exp \left\{ [A\Gamma(p)\zeta_+(1+p)z^{-p} - Z_p(0)\log z + Z_p'(0) + O(t^{c-})] \right\}, \quad (9)$$

where $0 < c_+, c_- < 1$ and $Z_p(s) \equiv Z_p \left| \frac{g}{h} \right| (s)$ is the p -dimensional Epstein zeta function which has a pole with residue A . In above equations $\zeta_-(s) \equiv \zeta_R(s)$ is

the Riemann zeta function, $\zeta_+(s) = (1 - 2^{1-s})\zeta_-(s)$. The total number of quantum states can be described by the quantities $r_{\pm}(N)$ defined by

$$K_{\pm}(t) = \sum_{N=0}^{\infty} r_{\pm}(N)t^N \equiv H_{\pm}(\Omega), \quad (10)$$

where $t = \exp(-z)$, $t < 1$, and N is a total quantum number. By means of the asymptotic expansion of $K_{\pm}(t)$ for $t \mapsto 1$, which is equivalent to expansion of $H_{\pm}(\Omega, 0)$ for small z and using the formulae (8) and (9) one arrives at complete asymptotic of $r_{\pm}(N)$. Thus for $N \mapsto \infty$ one has [6, 9]

$$r_{\pm}(N) = C_{\pm}(p)N^{(2Z_p(0)-p-2)/(2(1+p))} \times \\ \times \exp \left\{ \frac{1+p}{p} [A\Gamma(1+p)\zeta_{\pm}(1+p)]^{1/(1+p)} N^{p/(1+p)} \right\} [1 + O(N^{-\kappa_{\pm}})], \quad (11)$$

$$C_{\pm}(p) = [A\Gamma(1+p)\zeta_{\pm}(1+p)]^{(1-2qZ_p(0))/(2p+2)} \frac{\exp(Z'_p(0))}{[2\pi(1+p)]^{1/2}}, \quad (12)$$

$$\kappa_{\pm} = \frac{p}{1+p} \min \left(\frac{C_{\pm}(p)}{p} - \frac{\delta}{4}, \frac{1}{2} - \delta \right), \quad (13)$$

and $0 < \delta < \frac{2}{3}$.

4. Let us consider the membrane excitation states with non-trivial winding number around the target space torus. In this case the spectrum of the light-cone membrane Hamiltonian is discrete [8, 5]. The toroidal membrane is wrapped around the target space torus ($\mathcal{M} = \mathbf{S}^1 \otimes \mathbf{T}^2 \otimes \mathbf{R}^8$) and the winding number associated with this wrapping is $l_1 l_2$. For $l_1 l_2 \neq 0$ a membrane is topologically protected against usual supermembrane instabilities [10]. In the semiclassical approximation [8] the eleven-dimensional mass formula has the form ($d = 2$)

$$M^2(l) = (l_1 l_2 R_1 R_2)^2 + \mathcal{H}, \quad (14)$$

where the oscillator Hamiltonian can be written as follows

$$\mathcal{H} = 2 \sum_{\mathbf{n} \in \mathbf{Z}^2 / \{0\}} \left(\alpha_{\mathbf{n}}^{\dagger} \alpha_{\mathbf{n}} + \omega_{\mathbf{n}} \sigma_{\mathbf{n}}^{A\dagger} \sigma_{\mathbf{n}}^A \right), \quad (15)$$

in addition $\omega_{\mathbf{n}}^2 = \sum_j (l_j n_j R_j)^2$, and $A = 1, \dots, 8$ are $SO(7)$ spinor indices. The two constraints for the toroidal membrane are

$$C_j = l_j k_j + \mathcal{N}_{n_j}^{(b)} + \mathcal{N}_{n_j}^{(f)} = 0, \quad (16)$$

where $k_j = R_j p_j$ ($j = 1, 2$), p_j are discrete momenta. The commutation relations for above operators can be found, for example, in Ref. [8]. The Fock vacuum $|0\rangle$ and the mass of state (obtained by acting on the vacuum with creation operators) can be written correspondingly as follows

$$(\text{mass})^2 \sim \alpha_{n_1 n_2}^{\dagger} \dots \alpha_{n_1 n_2}^{\dagger} \sigma_{n_1 n_2}^{\dagger} \dots \sigma_{n_1 n_2}^{\dagger} |0\rangle. \quad (17)$$

As usual the physical Hilbert space consists of all Fock space states obeying the conditions (16). The quantized momenta p_1 and p_2 correspond to central charges in $N = 2$ nine-dimensional supersymmetry that classify the fluctuations about classical solution. The fact that the $M^2(l)$ is non-vanishing means that no multiplet-shortening will take place and so the vacuum must correspond to a long massive $N = 2$ multiplet with $M^2(l) = (l_1 l_2 R_1 R_2)^2$.

Let us demonstrate a correspondence between the semiclassical membrane and string results. If $R_2 \mapsto \infty$ while at the same time increasing l_2 such that the product $l_2 R_2$ is kept fixed ($l_2 R_2 = 1$, for example) then one dimension is shrunk while keeping the energy constant and producing a closed string. In this limit non-zero momentum p_2 is excluded and one obtains from Eq. (14) the mass formula for a ten-dimensional superstring compactified on a circle of radius R_1 . Thus the nine-dimensional mass is given by

$$M^2 = l_1^2 R_1^2 + \frac{k_1^2}{R_1^2} + 2 \sum_{j=1}^8 \sum_{n_1 \neq 0} \left(\alpha_{n_j}^\dagger \alpha_{n_j} + |n_1| \sigma_{n_j}^\dagger \sigma_{n_j} \right). \quad (18)$$

In this limit one of the constraints (16) becomes empty, while the other one yields

$$C_1 = l_1 k_1 + \sum_{j=1}^8 \sum_{n_1 \neq 0} \left(\text{sign}(n_1) \alpha_{n_j}^\dagger \alpha_{n_j} + n_1 \sigma_{n_j}^\dagger \sigma_{n_j} \right) = 0. \quad (19)$$

Note that in accordance with our definitions α_n^\dagger and σ_n^\dagger create left (right) - moving states when $n_1 > 0$ ($n_1 < 0$), while α_n and σ_n annihilate left (right) - moving states when $n_1 > 0$ ($n_1 < 0$). The constraint (19) is equivalent to the usual condition relating left and right Hamiltonians for the string.

Let us compare the BPS part of the membrane spectrum with the type II string BPS spectrum. The correspondence between spectra at zero-mode level was established in Ref. [1]. Using the constraint (19) in the mass formula (18) for free string states (note that under T -duality relating IIA and IIB spectra, $R_1 \mapsto \alpha' R_1^{-1}$, for the sake of simplicity here and in the following we assume an inverse string tension parameter α' equal to 1) we have

$$M^2 = (l_1 R_1 + k_1 / R_1)^2. \quad (20)$$

The Kaluza-Klein mass for perturbative or $(1, 0)$ string states with $k_2 = 0$ (zero Ramond-, Ramond charge) is given by

$$M_{IIB}^2 = \frac{k_1^2}{R_1^2} + l_1^2 R_1^2 + 2 \left(\mathcal{N}^{(b)} + \mathcal{N}^{(f)} \right). \quad (21)$$

For the general BPS perturbative states with the corresponding oscillating states $(1, 0)$ the masses given in formula (21) coincide with the masses of Eq. (20). The last statement is correct also for the non-perturbative type IIB string [2, 11] with charges (q_1, q_2) . In this case $k_1 = m q_1$, $k_2 = m q_2$, and q_1, q_2 being co-prime. For the NS-NS string we have $k_2 = 0$, $q_1 = 1$, $q_2 = 0$, while for the R-R string, $k_1 = 0$, $q_1 = 0$, $q_2 = 1$ and the value of M_{IIB}^2 coincides with masses (up to change of indices $1 \mapsto 2$) given in formula (20). Hence, for calculation of BPS membrane Hagedorn density one can use stringy results.

If $l_2 = 0$ (or $l_1 = 0$) then the stable classical solution will be collapsed to string-like membrane wound around only one compact direction in the target space. For this classical configuration the mass formula (14) indicates the presence of massless states. The world-volume metric for this light-cone classical solution is degenerate, but the field equations are nevertheless non-singular. It is not clear whether the semiclassical approximation can be trusted for such a configuration [8]. Indeed the equation $\omega_{n_1} = \sqrt{(n_1 l_1 R_1)^2}$ shows that all values of n give the same frequency ω_{n_1} in the semiclassical approximation. This infinite degeneracy will presumably be lifted when the higher order terms in the Hamiltonian are included.

5. Let us suppose that the semiclassical approach to quantization leads to a second quantized brane theory which can be considered as a theory of non-interacting strings. Then the Hilbert space of all multiple string states that satisfy the BPS conditions (zero branes) with a total energy momentum P has the form [12]

$$\mathcal{H}_P = \bigoplus_{\sum l N_l = N_P} \bigotimes_l \text{Sym}^{N_l} \mathcal{H}_l, \quad (22)$$

where symbol Sym^N indicates the N -th symmetric tensor product. The exact dimension of \mathcal{H}_P is determined by the character expansion formula

$$\sum_{N_P} \dim \mathcal{H}_P q^{N_P} \simeq \prod_l \left(\frac{1+q^l}{1-q^l} \right)^{\frac{1}{2} \dim \mathcal{H}_l}, \quad (23)$$

where the dimension \mathcal{H}_l of the Hilbert space of single string BPS states with momentum $k = l\hat{P}$ is given by $\dim \mathcal{H}_l = d(\frac{1}{2}l^2\hat{P})$, and $|\hat{P}|^2 = |\hat{P}_L|^2 - |\hat{P}_R|^2$ (see Ref. [12] for detail). The asymptotics of the generating function (23) and the dimension \mathcal{H}_P can be found with the help of Eqs. (11), (12) that is generalization of the Meinardus result for vector-valued functions.

The Eq. (23) is similar to the denominator formula of a (generalized) Kac-Moody algebra [13], which can be written as follows

$$\sum_{\sigma \in W} (\text{sgn}(\sigma)) e^{\sigma(\rho)} = e^\rho \prod_{r>0} (1 - e^r)^{\text{mult}(r)}, \quad (24)$$

where ρ is the Weyl vector, the sum on the left hand side is over all elements of the Weyl group W , the product on the right side runs over all positive roots (one has the usual notation of root spaces, positive roots, simple roots and Weyl group, associated with Kac-Moody algebra) and each term is weighted by the root multiplicity $\text{mult}(r)$. The Eq. (23) reduces to the standard superstring partition function for $\hat{P}^2 = 0$ [12]. The equivalent description of the second quantized string states on the five-brane can be obtained, for example, by considering the sigma model on the target space $\sum_N \text{Sym}^N T^4$. There is the correspondence between the formula (22) and the term at order $q^{\frac{1}{2}N_P \hat{P}^2}$ in the expansion of the elliptic genus of the orbifold $\text{Sym}^{N_P} T^4$ [12]. Using this correspondence one finds that the asymptotic growth is equal that of states at level $\frac{1}{2}N_P \hat{P}^2$ in a unitary conformal field theory with central charge proportional to N_P .

Our aim now is to compare the quantum states of a membrane and a black hole. The correspondence between asymptotic state density of the fundamental p -brane and the four-dimensional black hole for $p \rightarrow \infty$ was found in Refs. [14]. An extreme black hole states can be identified also with the highly excited states of a fundamental string. The relevant string states are BPS states, while the string entropy is identical to the Bekenstein-Hawking entropy defined by the horizon area [3]. For a calculation of the ground state degeneracy of systems with quantum numbers of certain BPS extreme black holes the D -brane method can be used. A typical 5-dimensional example has been analyzed in Refs. [15–17]. Working in the type IIB string theory on $M^5 \otimes T^5$ one can construct a D -brane configuration such that the corresponding supergravity solutions describe 5-dimensional black holes. In addition five branes and one branes are wrapped on T^5 and the system is given by Kaluza–Klein momentum N in one of the directions. Therefore the three independent charges (Q_1, Q_5, N) arise in the theory, where Q_1, Q_5 are electric and a magnetic charges respectively. The naive D -brane picture gives the entropy in

terms of partition function $H_{\pm}(z)$ for a gas of $Q_1 Q_5$ species of massless quanta. For $p=1$ the integers r_{\pm} in Eq. (11) represent the degeneracy of the state with momentum N . Thus for $N \mapsto \infty$ one has

$$\log r_{\pm}(N) = \sqrt{q\zeta_{\pm}(2)N} - \frac{q+3}{4}\log N + \log C_{\pm}(1) + O(N^{-\kappa_{\pm}}), \quad (25)$$

where

$$C_{+}(1) = 2^{-\frac{1}{2}} \left(\frac{q}{16}\right)^{(q+1)/2}, \quad C_{-}(1) = C_{+}(1) \left(\frac{4}{3}\right)^{(q+1)/2}. \quad (26)$$

For fixed $q=4Q_1 Q_5$ the entropy is given by

$$S = [\log(r_{+}(N)r_{-}(N))]_{N \mapsto \infty} \simeq c_1 \sqrt{Q_1 Q_5 N} - \log N (c_2 Q_1 Q_5 + c_3), \quad (27)$$

where c_j ($j=1,2,3$) are some positive numbers. This expression agrees with the classical black hole entropy.

Note that for $p > 1$, using the mass formula $M^2 = N$, for the number of branes states of mass M to $M + dM$ one can find (see also [14])

$$\varrho(M)dM \simeq 2C_{\pm}(p)M^{(p-1)/(p+1)} \exp \left[b_{\pm}(p)M^{2p/(p+1)} \right]. \quad (28)$$

$$b_{\pm}(p) = \left(1 + \frac{1}{p}\right) [A\Gamma(p+1)\zeta_{\pm}(p+1)]^{1/(p+1)}. \quad (29)$$

This result has a universal character for all p -branes.

Recently it has been pointed out that the classical result (27) is incorrect when the black hole becomes massive enough for its Schwarzschild radius to exceed any microscopic scale such as the compactification radii [16, 17]. Indeed, if the charges (Q_1, Q_5, N) tend to infinity in fixed proportion $Q_1 Q_5 = Q(N)$, then the correct formula does not agree with the black hole entropy (27). If, for example, $Q(N) = N$, then for $N \mapsto \infty$ one finds $\log[r_{+}(N)r_{-}(N)] \sim N \log N$. The naive D -brane prescription, therefore, fails to agree with U -duality which requires symmetry among charges (Q_1, Q_5, N) [17].

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1. J.H.Schwarz, Phys. Lett. **B 360**, 13 (1995).
 2. J.G.Russo and A.A.Tseytlin, hep-th/9611047 (1996).
 3. A.Strominger and C.Vafa, Phys. Lett. **B 379**, 99 (1996); G.T.Horowitz and A.Strominger, Phys. Rev. Lett. **77**, 2368 (1996); C.V.Johnson, R.R.Khuri, and R.C.Myers, Phys. Lett. **B 378**, 78 (1996).
 4. G.T.Horowitz and J.Polchinski, hep-th/9612146 (1996).
 5. A.A.Bytsenko and S.D.Odintsov, Phys. Lett. **B 245**, 21 (1990); A.A.Bytsenko and S.D.Odintsov, Fortschr. Phys. **41**, 233 (1993).
 6. E.Elizalde, S.D.Odintsov, A.Romeo et al., *Zeta Regularization Techniques with Applications*, World Sci., Singapore, 1994; A.A.Bytsenko, G.Cognola, L.Vanzo, and S.Zerbini, Phys. Reports **266**, No 1,2 (1996).
 7. D.Mumford, *Tata Lectures on Theta I, II*, Birkhäuser, 1983, 1984.
 8. M.J.Duff, T.Inami, C.N.Pope et al, Nucl. Phys. **B 297**, 515 (1988).
 9. A.A.Bytsenko and S.D.Odintov, hep-th/9611151 (1996).
 10. B.de Wit, M.Lüscher, and H.Nicolai, Nucl. Phys. **B 320**, 135 (1989).
 11. J.G.Russo, hep-th/9701188 (1997).
 12. R.Dijkgraaf, E.Verlinde, and H.Verlinde, hep-th/9603126 (1996).
 13. R.E.Borcherds, Invent. Math. **120**, 161 (1995); J.A.Harvey and G.Moore, Nucl. Phys. **B 463**, 315 (1996).
 14. A.A.Bytsenko, K.Kirsten, and S.Zerbini, Phys. Lett. **B 304**, 235 (1993); A.A.Bytsenko, K.Kirsten, and S.Zerbini, Mod. Phys. Lett. **A 9**, 1569 (1994).
 15. C.Callan and J.Maldacena, Nucl. Phys. **B 475**, 645 (1996).
 16. J.Maldacena and L.Susskind, Nucl. Phys. **B 475**, 679 (1996).
 17. E.Halyo, A.Rajaraman, and L.Susskind, hep-th/9605112 (1996).