РОССИЙСКАЯ АКАДЕМИЯ НАУК

ПИСЬМА В ЖУРНАЛ ЭКСПЕРИМЕНТАЛЬНОЙ И ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

ОСНОВАН В 1965 ГОДУ *ТОМ 66, ВЫПУСК 7*
ВЫХОДИТ 24 РАЗА В ГОД *10 ОКТЯБРЯ, 1997*

Pis'ma v ZhETF, vol.66, iss.7, pp.449 - 453

ORIGIN OF A CLASSICAL SPACE IN QUANTUM INHOMOGENEOUS MODELS

A.A.Kirillov[1)], *G.Montani**

Institute for Applied Mathematics and Cybernetics
603005 Nizhny Novgorod, Russia

**ICRA Dipartimento di Fisica Universitá di Roma*
I-00185 Roma, Italy

Submitted 28 August 1997

The origin of a classical background geometry in quantum vacuum inhomogeneous master model is investigated. It is shown that the background appears at the moment when the horizon size has the order of the characteristic scale of inhomogeneity of the universe and the local anisotropy can be described by small perturbations.

PACS: 98.80.Hw

It is widely recognized that the most realistic models of the early universe have to contain an inflationary epoch [1, 2]. Such an epoch is known to have the energy scales $H = \dot{a}/a \sim 10^{-5} m_{pl}$ much below the Planck energy and at these scales the spacetime is believed to have a classical nature. In particular, this represents the main base for the semiclassical description of the early universe (e.g., see Ref. [3] and references therein). However the quantum boundary (the moment of origin of a stable classical background geometry) is not so well defined as it is commonly believed now. In general the moment when the spacetime acquires a semiclassical nature has to depend upon initial conditions and represents, in quantum cosmology, a free parameter.

We note that the problem of the origin of a classical background does not coincide with the problem of the quasi-classical limit in quantum gravity and cosmology. Indeed, quantum non-linear inhomogeneous gravitational fields near the singularity was shown to be described by stationary states [4] (see also the multidimensional case in Refs. [5]). Therefore, there is no difficulty to construct high frequency wave packets and to consider the quasi-classical limit in spite the fact how close to the singularity the universe is.

1) e-mail: kirillov@focus.nnov.su

However the background geometry was shown to be absent near the singularity (it tuns out to be hidden under pure quantum fluctuations [4]). In fact the moment of the origin of a background depends upon the particular choice of an initial quantum state and, therefore, appears as a free parameter. In the present paper we study the origin of a classical background in a vacuum long-wave asymptotic models [6] $L_i \gg L_h$, where L_i, L_h are a characteristic scale of inhomogeneity of the gravitational field and the horizon size respectively (we note that in quantum gravity under the horizon size and the scale of inhomogeneity one should understand corresponding mean values) and show that the stable background appears at the moment when the horizon size reaches the scale of the inhomogeneity $L_i \sim L_h$. Such an estimate remains also valid in the quasi-classical limit. In chaotic inflationary scenarios [2, 7] this means that the inflationary epoch starts when the semiclassical approximation is not valid. We consider the ADM (Arnowitt – Deser – Misner) scheme of quantization [8], since in this case we avoid the problem of the probabilistic interpretation when considering the matter sources and, besides, this scheme represents the only rigorous way to treat the case when a part of the universe can recollapse [3].

Near the singularity the behaviour of the long-wave inhomogeneous gravitational field can be described in the approximation of deep oscillations [4, 6] as follows. The metric tensor has the representation in the Kasner-like form

$$ds^2 = N^2 dt^2 - R^2 \sum_{a=1}^{3} \exp\{q^a\} \left(\ell^a_\alpha dx^\alpha + N^a dt\right)^2, \tag{1}$$

where $\ell^a_\alpha(x)$ are Kasner vectors ($\det \ell^a_\alpha = 1$) and we distinguished a slow function of time R which characterizes the absolute value of the metric functions [9, 10] and is specified by initial conditions (see below). Near the singularity it is convenient to make use of the following parametrization of the scale functions [6]

$$q^a = Q_a \ln g; \ \sum Q_a = 1, \tag{2}$$

where the anisotropy parameters Q_q and $\ln g = \sum q^a$ can be expressed in terms of a new set of variables τ, y^i $(i = 1, 2)$, as follows

$$Q_a(y) = \frac{1}{3}\left(1 + \frac{2y^i A^a_i}{1+y^2}\right), \quad \ln g = -3e^{-\tau}\frac{1+y^2}{1-y^2}, \tag{3}$$

where A^a_i is a constant matrix, e.g., see in Ref. [6]. The parametrization (3) has the range $y^2 < 1$ and $-\infty < \tau < \infty$, $(0 \le g \le 1)$ and an appropriate choice of the function R allows to cover, by this parametrization, all of the classically allowed region of the configuration space.

The evolution (rotation) of Kasner vectors is completely determined by the momentum constraints [11], while the evolution of scale functions is described by the action (we use Planck's units $m_{pl} = 16\pi$, see Refs. [4, 6], and fix the gauge $N^a = 0$)

$$I = \int \{(\mathbf{P}\frac{\partial \mathbf{y}}{\partial t} + h\frac{\partial \tau}{\partial t}) - \frac{N}{6R^3\sqrt{g}} e^{2\tau}[\varepsilon^2 - h^2 + 6e^{-2\tau}V(\tau, \mathbf{y})]\} d^3x dt, \tag{4}$$

where $\varepsilon^2 = 1/4\left(1-y^2\right)^2 \mathbf{P}^2$ and the potential term $V = R^6 g\left(-{}^3\mathrm{R}\right)$ (${}^3\mathrm{R}$ is the scalar curvature with the metric (1)) has the following decomposition

$$V = R^4 \sum_{A=1}^{k} \lambda_A g^{\sigma_A}, \tag{5}$$

where coefficients λ_A represent functions of all dynamical variables (and slow functions of $\ln g$) which characterize an initial degree of inhomogeneity of the gravitational field and $\sigma_{abc} = 1 + Q_a - Q_b - Q_c$, $b \neq c$. In the approximation of deep oscillations $g \ll 1$ this potential can be modeled by a set of potential walls

$$g^{\sigma_A} \to \theta_\infty[\sigma_A(Q)] = \begin{cases} +\infty\,, & \sigma_A < 0, \\ 0\,, & \sigma_A > 0, \end{cases} \tag{6}$$

and is independent of Kasner vectors $V_\infty = \sum \theta_\infty\left(Q_a\right)$.

By solving the Hamiltonian constraint $H = 0$ in (4) we define the ADM action reduced to the physical sector as follows

$$I = \int (\mathbf{P_y} \cdot \frac{d\mathbf{y}}{d\tau} - H_{\mathrm{ADM}}) d^3x d\tau, \tag{7}$$

where $H_{\mathrm{ADM}} \equiv -h = \pm\sqrt{\varepsilon^2 + 6\,e^{-2\tau}V}$ is the ADM energy density and τ plays the role of time ($\dot{\tau} = 1$) which corresponds to the gauge $N_{\mathrm{ADM}} = (3R^3\sqrt{g}/H_{\mathrm{ADM}})e^{-2\tau}$. The sign of H_{ADM} depends upon initial conditions and is determined from the requirement that H_{ADM} represents a differentiable function of coordinates (the positive sign of H_{ADM} corresponds to expanding regions of the space).

The condition of applicability of the approximation (6) can be written as follows

$$\varepsilon^2 \gg 6e^{-2\tau}V \tag{8}$$

as $Q_a > \delta > 0$ ($\delta \ll 1$). Thus, from the condition that the approximation of deep oscillations (6) breaks at the moment $g \sim 1$, one finds that the function R should be chosen as follows $R^4 = (\varepsilon^2/6\lambda)e^{2\tau}$ (where $\lambda = |\sum \lambda_A|$) and the inequality (8) reads $g \ll 1$.

The synchronous cosmological time relates to τ by means of the equation $dt = N_{\mathrm{ADM}} d\tau$ from which we find the estimate $\sqrt{g} \sim t/t_0$, where $t_0 = cL_i\sqrt{\varepsilon L_i}$, $L_i^2 \sim 1/\lambda$ is a characteristic scale of the inhomogeneity, ε is the ADM energy density ($\varepsilon = \mathrm{const}$), and c is a slow (logarithmic) function of time ($c \sim 1$ as $g \to 1$). Thus, in the synchronous time the upper limit of the approximation (6) is $t \sim t_0$. We note that from the physical viewpoint t_0 corresponds to the moment when the horizon size reaches the characteristic scale of inhomogeneity and both terms in the Hamiltonian constraint (the kinetic and potential terms) have the same order.

The physical sector of the configuration space (variables $\mathbf{y}$) is a realization of the Lobachevsky plane and the potential V_∞ limits the part $K = \{Q_a \geq 0\}$. Quantization of this system can be carried out as follows. The ADM density of energy represents a constant of motion and, therefore, we can define stationary states as solutions to the eigenvalue problem for the set of Laplace - Beltrami operators $\varepsilon^2\left(x\right) = \Delta\left(x\right) + 1/4P\left(x\right)$ (see Refs. [4, 5])

$$(\Delta + k_n^2 + \frac{1}{4}P)\varphi_n(y) = 0, \qquad \varphi_n\mid_{\partial K} = 0, \tag{9}$$

where the Laplace operator Δ is constructed via the metric $\delta l^2 = h_{ij}\delta y^i \delta y^j = r^2(4(\delta y)^2/(1-y^2)^2)$, r and P are determined through a renormalization procedure (in the discrete approximation one can define $r = \left(\Delta x\right)^3$, $P = 1/r^2 + 4k_0^2$, so that in the ground state $_0 = 0$). The eigenstates φ_n are classified by the integer-valued function $n\left(x\right)$ and obey the orthogonality and normalization relations

$$(\varphi_n, \varphi_m) = \int_K \varphi_n^*\left(y\right)\varphi_m(y) D\mu(y) = \delta_{nm}, \tag{10}$$

where

$$D\mu(y)=\prod_x \frac{1}{\pi}\sqrt{h}d^2y\,(x)\,,\quad K=\prod_x K\,(x)$$

, and π is the volume of $K\,(x)$. Thus, an arbitrary solution Ψ to the Shredinger equation $i\partial_\tau\Psi = H_{ADM}\Psi$ takes the form

$$\Psi=\sum_n A_n \exp(-iH_n\tau)\varphi_n(y), \tag{11}$$

where $H_n = \int k_n d^3x$ and A_n are arbitrary constants ($\sum_n \mid A_n \mid^2 = 1$) which are to be specified by initial conditions. The probabilistic distribution for variables y has the standard form $P\,(y,\tau) = |\Psi\,(y,\tau)|^2$ which coincides with that one in Ref. [5] derived on the base of the Newton-Wigner states. The function $n\,(x)$ plays the role of filling numbers for frozen non-linear gravitational waves whose wave-lengths exceed the horizon size (the density of excitations for the local anisotropy [4, 12]). The eigenstates φ_n define stationary (in terms of the anisotropy parameters $Q\,(y)$) quantum states and describe, in the case of $H_n > 0$, an expanding universe with a fixed energy density of the anisotropy.

For an arbitrary quantum state Ψ one can determine the background metric $\langle ds^2\rangle$. However such a background is stable and has sense only when quantum fluctuations around it are small. In the case of $g \ll 1$ fluctuations well exceed the average metric and the background is hidden [4]. Indeed in this case for the moments of scale functions one can find the estimate (in the same way as in Ref. [5])

$$\langle a_i^m\rangle = \langle R^m g^{\frac{m}{2}Q_i}\rangle \sim D_i\,(m,\tau)\exp\left(\frac{m+5}{2}\tau\right), \tag{12}$$

where the function $D_i\,(m,\tau)$ depends upon the choice of the initial quantum state.

Consider now an arbitrary stationary state φ_n which gives the stationary probabilistic distribution $P\,(y) = |\varphi_n|^2$. In this case $D = bk_n{}^{m/2}\left(L_i^{m/2}\right)_n$ is a constant, where the characteristic scale of inhomogeneity $\left(L_i^{m/2}\right)_n$ is determined via the momentum constraints and b comes from the uncertainty in the operator ordering. Thus, for the intensity of quantum fluctuations one finds the divergent, in the limit $g \to 0$ ($\tau \to -\infty$), expression $\langle\delta^2\rangle = \left(\langle a^2\rangle / \langle a\rangle^2 - 1\right) \sim e^{-5/2\tau}$ which explicitly shows the instability of the average geometry as $g \ll 1$. The intensity of quantum fluctuations reaches the order $\delta \sim 1$ at the moment $t \sim \left(L_i^{3/2}\right)_n \sqrt{k_n}$ ($g \sim 1$) when the anisotropy functions can be described by small perturbations $a_i^2 = R^2\,(1 + Q_i \ln g + \ldots)$ and the universe acquires a quasi-isotropic character. This moment can be considered as the moment of the origin of a stable classical background.

Consider now the case of quasi-classical states. Classical trajectories of this system were shown to have a chaotic behaviour [6] and this leads to an additional quick spreading of an arbitrary initial wave packet (e.g., see Ref. [13] and references therein). An arbitrary initial quantum uncertainty increases as $\Delta\Gamma \sim \Delta\Gamma_0 e^S$, where $S = (\tau - \tau_0)$ is the geodesic path, and quickly reaches the maximum value $\Gamma_{max} \sim 2\pi^2\,\langle H\rangle$ (where Γ_{max} is the phase volume)[2]. Thus the picture in which the center of the wave packet traces a classical trajectory remains valid during the period $\Delta\tau = \ln(\Gamma_{max}/\Delta\Gamma_0)$. In quantum theory the

[2] When $\Delta\Gamma \sim \Gamma_{max}$ the exponential behaviour is replaced by a power law [13], however this does not significantly change all the subsequent estimates.

minimum value $\Delta\Gamma_0 \sim 1$ and hence one finds $\Delta\tau_{max} \sim \ln \bar{n}$ where $\bar{n}$ is the average density of excitations corresponding to the energy $\langle H \rangle$. After the period $\Delta\tau_{max}$ the function $D_i(m,\tau)$ in (12) becomes almost constant and we have the situation described earlier.

We point out to the fact that the period $\Delta\tau_{\max}$ can be arbitrary large (that depends upon a particular choice of initial conditions). This means that in the problem of a cosmological collapse the picture in which the universe is a classical object can be valid up to arbitrary small times $t \ll t_{pl}$. However in the problem of the cosmological expansion the universe spends infinite period of time τ starting from the singularity and, therefore, an arbitrary initial wave packet will spread over the whole configuration space and the classical picture is invalid.

In conclusion we stress that in the presence of matter the quantum evolution of the inhomogeneous universe requires a separate consideration, however the general fact is that the stable background appears at the moment when the local anisotropy can be considered as perturbations.

This research was supported in part (for A.A.K) by the Russian Foundation for Basic Research under the grant 95-02-04935 and the Russian research project "Cosmomicrophysics".

---

1. A.A.Starobinsky, Phys. Lett. **91B**, 100 (1980); A.H. Guth, Phys. Rev. **D23**, 347 (1981); A.A. Linde, Phys. Lett. **B108**, 389 (1982).
2. A.D.Linde, *Particle Physics and Inflationary Cosmology*, Harwood Academic, 1990.
3. A.O.Barvinsky, Phys. Rep. **230**, 237 (1993); B.L. Al'tshuler and A.O.Barvinsky, Usp. Fiz. Nauk **166**, 459 (1996).
4. A.A.Kirillov, Phys. Lett. **B 399**, 201 (1997); Int. Jour. Mod. Phys. **D3** , 431 (1994).
5. A.A.Kirillov, Pis'ma ZhETF **62**, 81 (1995); A.A.Kirillov and V.N. Melnikov, Phys. Lett. **B389**, 221 (1996).
6. A.A.Kirillov, ZhETF **103**, 721 (1993). [Sov. Phys. JETP **76**, 355 (1993)]; A.A. Kirillov and V.N. Melnikov, Phys.Rev. **D51**, 723 (1995); A.A. Kirillov and G. Montani, Phys. Rev. **D** (1997), in press.
7. A.D.Linde, Phys. Lett. **B129**, 177 (1983); A.A. Starobinsky, in: *Current Topics in Field Theory, Quantum Gravity and Strings*, Lecture Notes in Physics, Eds. H.J. de Vega and N. Sanchez, Springer-Verlag, Heidelberg, 1986, **246**, p.107.
8. R.Arnowitt, S. Deser, and C.W. Misner, in: Gravitation: *An Introduction to Current Research*, Ed. L.Witten, New York: Wiley, 1962, p.227.
9. C.W.Misner, Phys. Rev. Lett. **22**, 1071 (1969).
10. C.W.Misner, K.S.Thorne and J.A. Wheeler, *Gravitation*, San Francisco: Freeman, 1973, vol.2.
11. V.A.Belinskii, E.M.Lifshitz and I.M.Khalatnikov, Adv. Phys. **31**, 639 (1982); ZhETF **62**, 1606 (1972) [Sov. Phys. JETP **35**, 838 (1972)].
12. C.W.Misner, Phys. Rev. **186**, 1319 (1969); J.A.Wheeler, in:*Magic Without Magic*, Ed. J.Klander, San Francisco: Freeman, 1972.
13. P.V.Elyutin, Usp. Fiz. Nauk **155**, 397 (1988).