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ZERO CURVATURE REPRESENTATION FOR CLASSICAL LATTICE SINE-GORDON MODEL VIA QUANTUM R-MATRIX

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Local M-operators for the classical sine-Gordon model in discrete space-time are constructed by convolution of the quantum trigonometric 4×4 R-matrix with certain vectors in its "quantum" space. Components of the vectors are τ-functions of the model. This construction generalizes the known representation of continuous time M-operators through classical r-matrix.

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1. Soliton equations are integrable hamiltonian systems [1], with Poisson brackets for Lax matrices having a unified form in terms of (classical) r-matrix. An alternative approach [1], [2] consists in representing soliton equations as 2D zero curvature (ZC) conditions for a pair of matrices called L and M-operators depending on a spectral parameter. Although this method avoids any reference to the hamiltonian aspects, the r-matrix arise here, too, as a machine to produce M-operators from L-operators. Let us recall how it works.

Let $\mathcal{L}_l(z)$ be a classical ultralocal 2×2 L-operator on 1D lattice with the periodic boundary condition $\mathcal{L}_{l+N}(z) = \mathcal{L}_l(z)$; z is the spectral parameter. The monodromy matrix is $\mathcal{T}_l(z) = \mathcal{L}_{l+N-1}(z)\dots\mathcal{L}_{l+1}(z)\mathcal{L}_l(z)$. Hamiltonians of commuting flows are obtained by expanding $\log T(z)$ in z, where $T(z) = \mathrm{tr}\,\mathcal{T}_l(z)$ does not depend on l due to the periodic boundary condition. All these flows admit a ZC representation. The generating function of corresponding M-operators is [3, 1]

$$M_l(z;w) = T^{-1}(w)\,\mathrm{tr}_1\left[r(z/w)(\mathcal{T}_l(w)\otimes I)\right], \tag{1}$$

where $r(z)$ is the r-matrix (of size 4×4) acting in the tensor product of two 2-dimensional spaces, tr_1 means trace in the first space, I is the unity matrix.

A way to construct local M-operators from (1) is well known [4, 1, 5]. Suppose there exists z_0 such that $\det\mathcal{L}_l(z_0) = 0$ for any l, so $\mathcal{L}_l(z_0)$ is a projector:

$$\mathcal{L}_l(z_0) = \frac{|\alpha_l\rangle\langle\beta_l|}{\lambda_l}, \qquad |\alpha\rangle = \begin{pmatrix}\alpha^{(1)}\\ \alpha^{(2)}\end{pmatrix}, \qquad \langle\beta| = \left(\beta^{(1)}, \beta^{(2)}\right). \tag{2}$$

Here λ_l is a scalar normalization factor. Then $M_l(z;z_0)$ is a local quantity:

$$M_l(z) \equiv M_l(z;z_0) = \frac{\langle\beta_l|r(z/z_0)|\alpha_{l-1}\rangle}{\langle\beta_l|\alpha_{l-1}\rangle}. \tag{3}$$

The scalar product is taken in the first space only, so the result is a 2×2 matrix. It obeys the ZC condition $\partial_t\mathcal{L}_l(z) = M_{l+1}(z)\mathcal{L}_l(z) - \mathcal{L}_l(z)M_l(z)$ with the spectral parameter.

The goal of this work is to extend eq. (3) to M-operators for *discrete time flows* in Hirota's 2D partial difference equations [6–8]. We follow [9,10], treating the discrete equations as members of the same infinite hierarchy as the continuous ones.

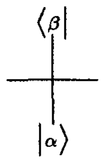
Let us outline the results. In the discrete case $r(z)$ in (3) is substituted by quantum R-matrix. Specifically, the following representation of discrete M-operators $\mathcal{M}_l(z)$ holds:

$$\mathcal{M}_l(z) = \frac{\langle\beta_l|R(z/z_0)|\check{\beta}_{l-1}\rangle}{\langle\beta_l|\alpha_{l-1}\rangle}\,, \qquad |\check{\beta}_l\rangle \equiv \sigma_1|\beta_l\rangle \tag{4}$$

(hereafter σ_i are Pauli matrices). In the r.h.s., $R(z)$ is a *quantum* 4×4 R-matrix to be specified below with the "quantum" parameter q related to the time lattice spacing. A similar formula for the L-operator itself is valid with another quantum R-matrix $R^{(-)}(z)$:

$$\mathcal{L}_l(z) = \frac{\langle\beta_l|R^{(-)}(z/z_0)|\alpha_l\rangle}{\langle\beta_l|\alpha_{l-1}\rangle}\,. \tag{5}$$

The vectors $|\alpha_l\rangle$ and $|\beta_l\rangle$ are *the same* as in eq. (3). In the language of the algebraic Bethe ansatz [11, 3], the scalar product is taken in the "quantum" (vertical) space, so one gets a 2×2 matrix in the "auxiliary" (horizontal) space:

$\langle\beta|R(z)|\alpha\rangle =$ 

The M-operator (4) generates shifts of a time variable m. The ZC condition

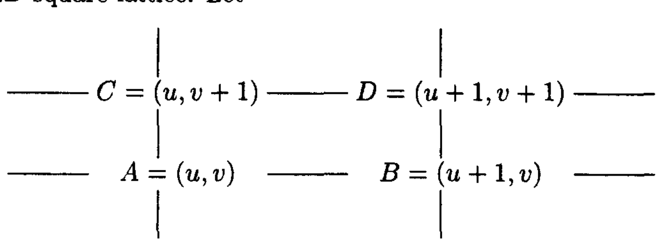
$$\mathcal{M}_{l+1,m}(z)\mathcal{L}_{l,m}(z) = \mathcal{L}_{l,m+1}(z)\mathcal{M}_{l,m}(z) \tag{6}$$

gives rise to the discrete soliton equations from [6, 8].

The change of dynamical variables to the pair of vectors $|\alpha_l\rangle$, $|\beta_l\rangle$ plays a key role. Using equations of motion of the discrete model, we show that (suitably normalized) components of the vectors $|\alpha_l\rangle$, $|\beta_l\rangle$ are τ-*functions* (on τ-functions see e.g. [12]).

In this paper we elaborate the simplest example – the lattice sine-Gordon (SG) model. There are two lattice versions of the classical SG model: the model on a space lattice with continuous time [5, 13] and Hirota's SG equation on a space-time lattice [8]. They have common L-operator. The M-operators are given by (3) with the trigonometric classical r-matrix for the former and (4) for the latter, with $R(z)$ being the simplest trigonometric solution of the quantum Yang – Baxter equation (the R-matrix of the XXZ spin chain).

2. By the SG model on a space-time lattice we mean the Faddeev – Volkov version [14, 15] of Hirota's discrete SG equation [8]. This is a non-linear equation for a function $\psi(u, v)$ on the 2D square lattice. Let



be an elementary cell of the u,v-lattice. In this notation the equation reads

$$\nu\psi_C\psi_D - \nu\psi_A\psi_B = \mu\big(\psi_B\psi_D - \mu\psi_A\psi_C\big)\,, \tag{7}$$

where μ,ν are constants. It contains both KdV and SG equations as different continuum limits. Eq. (7) can be represented [14] as the ZC condition $L_{D\leftarrow B}(z;\nu)L_{B\leftarrow A}(z;\mu) = L_{D\leftarrow C}(z;\mu)L_{C\leftarrow A}(z;\nu)$ with the L-matrix [14, 16]

$$L_{B\leftarrow A}(z;\mu) = \begin{pmatrix} \mu\psi_B^{\frac12}\psi_A^{-\frac12} & z\psi_B^{-\frac12}\psi_A^{-\frac12} \\ z\psi_B^{\frac12}\psi_A^{\frac12} & \mu\psi_B^{-\frac12}\psi_A^{\frac12} \end{pmatrix}. \tag{8}$$

We call $l = \frac12(u+v)$, $m = \frac12(u-v)$ discrete space and time coordinates respectively. Consider "composite" L and M-operators generating shifts $A \to D$ and $C \to B$ respectively: $\hat{\mathcal{L}}_{D\leftarrow A}(z) = z^{-1}L_{D\leftarrow C}(z;\mu)L_{C\leftarrow A}(z;\nu)$, $\hat{\mathcal{M}}_{B\leftarrow C}(z) = z^{-1}(z^2-\nu^2)L_{B\leftarrow A}(z;\mu)\,[L_{C\leftarrow A}(z;\nu)]^{-1}$. From (8) we find:

$$\hat{\mathcal{L}}_{D\leftarrow A}(\mu z) = \begin{pmatrix} \mu z\psi_A^{\frac12}\psi_D^{-\frac12} + \nu z^{-1}\psi_D^{\frac12}\psi_A^{-\frac12} & \psi_C^{-1}\left(\mu\psi_D^{\frac12}\psi_A^{-\frac12} + \nu\psi_A^{\frac12}\psi_D^{-\frac12}\right) \\ \psi_C\left(\mu\psi_A^{\frac12}\psi_D^{-\frac12} + \nu\psi_D^{\frac12}\psi_A^{-\frac12}\right) & \mu z\psi_D^{\frac12}\psi_A^{-\frac12} + \nu z^{-1}\psi_A^{\frac12}\psi_D^{-\frac12} \end{pmatrix}, \tag{9}$$

$$\hat{\mathcal{M}}_{B\leftarrow C}(\mu z) = \begin{pmatrix} \mu z\psi_C^{\frac12}\psi_B^{-\frac12} - \nu z^{-1}\psi_B^{\frac12}\psi_C^{-\frac12} & \psi_A^{-1}\left(\mu\psi_B^{\frac12}\psi_C^{-\frac12} - \nu\psi_C^{\frac12}\psi_B^{-\frac12}\right) \\ \psi_A\left(\mu\psi_C^{\frac12}\psi_B^{-\frac12} - \nu\psi_B^{\frac12}\psi_C^{-\frac12}\right) & \mu z\psi_B^{\frac12}\psi_C^{-\frac12} - \nu z^{-1}\psi_C^{\frac12}\psi_B^{-\frac12} \end{pmatrix}. \tag{10}$$

The L-operator of the lattice SG model with continuous time [5] at l-th site is[1)]

$$\hat{\mathcal{L}}_l^{(IK)}(z) = \begin{pmatrix} z\chi_l + z^{-1}\chi_l^{-1} & s^{-\frac12}\varphi_l\pi_l \\ s^{-\frac12}\varphi_l\pi_l^{-1} & z\chi_l^{-1} + z^{-1}\chi_l \end{pmatrix}. \tag{11}$$

Here π_l, χ_l are exponentiated canonical variables, $\varphi_l = \left[1 + s(\chi_l^2 + \chi_l^{-2})\right]^{1/2}$, s is a parameter. To identify the L-operators (11) and (9), consider composite fields $\pi(u,v) = \psi^{1/2}(u+1,v)\psi^{1/2}(u,v+1)$, $\chi(u,v) = \psi^{1/2}(u,v)\psi^{-1/2}(u+1,v+1)$ and set $\pi_l = \pi(l,l)$, $\chi_l = \chi(l,l)$ at the constant time slice $m=0$. Identifying $s = \mu\nu(\mu^2+\nu^2)^{-1}$ and using eq. (7), we conclude that $\hat{\mathcal{L}}_l^{(IK)}(z) = (\mu\nu)^{-1/2}\hat{\mathcal{L}}_l\big((\mu\nu)^{1/2}z\big)$. Here $\hat{\mathcal{L}}_l(z) \equiv \hat{\mathcal{L}}_{D_l\leftarrow A_l}(z)$ where $A_l = (l,l)$, $D_l = (l+1,l+1)$. Similarly, we write $\hat{\mathcal{M}}_{\bar{B}_l\leftarrow A_l}(z) \equiv \hat{\mathcal{M}}_l(z)$, where $\bar{B}_l = (l+1,l-1)$. Then the discrete ZC condition acquires the form (6). The L-operator $\hat{\mathcal{L}}_l^{(IK)}(z)$ has two degeneracy points $z_0^{\pm} = (\mu/\nu)^{\pm\frac12}$ at which it is a projector (2) with the r.h.s. expressed through the field $\psi(u,v)$.

3. The idea of Hirota's approach [7] is to treat eq. (7) as a consequence of 3-term bilinear equations for τ-functions (see also [10, 17]). In the case at hand we need two τ-functions: τ and $\hat{\tau}$. Set

$$\psi(u,v) = \frac{\hat{\tau}(u,v)}{\tau(u,v)}\,, \tag{12}$$

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1) We take the L-operator from [5] and multiply it by σ_2 from the left to deal with eq. (7) rather than Hirota's equation.

then eq. (7) follows from

$$(\nu-\mu)\hat{\tau}_A\tau_D=\nu\tau_B\hat{\tau}_C-\mu\hat{\tau}_B\tau_C\,,\quad (\nu-\mu)\tau_A\hat{\tau}_D=\nu\hat{\tau}_B\tau_C-\mu\tau_B\hat{\tau}_C\,. \tag{13}$$

The equivalent form of these equations,

$$(\nu+\mu)\tau_B\hat{\tau}_C=\mu\tau_A\hat{\tau}_D+\nu\hat{\tau}_A\tau_D\,,\quad (\nu+\mu)\hat{\tau}_B\tau_C=\mu\hat{\tau}_A\tau_D+\nu\tau_A\hat{\tau}_D\,, \tag{14}$$

is equally useful. At last, we point out the relation

$$\tau(u-1,v)\hat{\tau}(u+1,v)+\hat{\tau}(u-1,v)\tau(u+1,v)=2\tau(u,v)\hat{\tau}(u,v)\,. \tag{15}$$

A few remarks are in order. Eqs. (13) form a part of the 2-reduced 2D Toda lattice hierarchy [18], where μ, ν are *Miwa's variables* [9]. They play the role of inverse lattice spacings for the elementary discrete flows u, v. Lattice spacing in the m-direction is then $(\mu\nu)^{-1}(\mu-\nu)$. Note that the u, v-coordinate axes are in general not orthogonal to each other. In particular, as it is seen from eqs. (13), at $\mu=\nu$ one must *identify* u with v, so the 2D lattice collapses to a 1D one. In this sense (15) follows from (14) at $\nu=\mu$.

4. We are ready to represent the M operator as a convolution of quantum R-matrix with some vectors in its "quantum" space. Consider quantum R-matrices

$$R^{(\pm)}(z;q)=(a(z)\pm b(z))I\otimes I+(a(z)\mp b(z))\sigma_3\otimes\sigma_3+c(\sigma_1\otimes\sigma_1+\sigma_2\otimes\sigma_2)\,, \tag{16}$$

where $a(z)=qz-q^{-1}z^{-1}$, $b(z)=z-z^{-1}$, $c=q-q^{-1}$, q is a "quantum" parameter and z is the spectral parameter. The R-matrices $R^{(+)}$ and $R^{(-)}$ differ by Drinfeld's twist. Both of them satisfy the quantum Yang–Baxter equation (in Sect. 1 $R(z)=R^{(+)}(z;q)$).

Let $|\alpha\rangle$, $|\beta\rangle$ be two vectors (see (2)) from the first ("quantum") space. Consider the convolution $\langle\beta|R^{(\pm)}(z;q)|\alpha\rangle$ in the first space. This is a 2×2 matrix in the second ("auxiliary") space:

$$\langle\beta|R^{(\pm)}(z;q)|\alpha\rangle=\begin{pmatrix}\beta^{(1)}\alpha^{(1)}a(z)\pm\beta^{(2)}\alpha^{(2)}b(z) & \beta^{(2)}\alpha^{(1)}c(z)\\ \beta^{(1)}\alpha^{(2)}c(z) & \pm\beta^{(1)}\alpha^{(1)}b(z)+\beta^{(2)}\alpha^{(2)}a(z)\end{pmatrix}. \tag{17}$$

Let us compare this with r.h.s. of eqs. (9), (10). To do that, we write elements of the L and M-operators in terms of the τ-functions (12) and after that use eqs. (13), (14) when necessary. The best result is achieved after the simple gauge transformation

$$\mathcal{L}_{A\leftarrow D}(z)=\left(\frac{\tau_D\hat{\tau}_D}{\tau_A\hat{\tau}_A}\right)^{1/2}\hat{\mathcal{L}}_{D\leftarrow A}(z)\,,\quad \mathcal{M}_{B\leftarrow C}(z)=\left(\frac{\tau_B\hat{\tau}_B}{\tau_C\hat{\tau}_C}\right)^{1/2}\hat{\mathcal{M}}_{B\leftarrow C}(z)\,. \tag{18}$$

Omitting details, we present the final result. Set $\langle\alpha|=(\tau,\,\hat{\tau})$, $\langle\beta|=(\hat{\tau},\,\tau)$, $q=\mu/\nu$. At the slice $m=0$ we have

$$\mathcal{L}_l(\mu z)=\frac{2\mu\nu}{\mu-\nu}\,\frac{\langle\beta_l|R^{(-)}(z;q)|\alpha_l\rangle}{\langle\beta_l|\alpha_{l-1}\rangle}\,,\quad \mathcal{M}_l(\mu z)=\frac{2\mu\nu}{\mu+\nu}\,\frac{\langle\beta_l|R^{(+)}(z;q)|\check{\beta}_{l-1}\rangle}{\langle\beta_l|\alpha_{l-1}\rangle}\,, \tag{19}$$

where the notation from the end of Sect.2 is used. Up to the constant prefactors these formulas coincide with the ones announced in Sect. 1. Location of the vectors

$$|\alpha_l\rangle=\begin{pmatrix}\tau(l,l+1)\\ \hat{\tau}(l,l+1)\end{pmatrix},\qquad |\beta_l\rangle=\begin{pmatrix}\hat{\tau}(l+1,l)\\ \tau(l+1,l)\end{pmatrix}. \tag{20}$$

is shown in the fig.1:

The normalization factor in eq. (2) is equal to $\lambda_l = \mu\nu(\mu-\nu)^{-1}\tau(l,l)\hat{\tau}(l,l)$.

5. At last we show that the r-matrix formula (3) is a degenerate case of eq. (4). A naive continuous time limit would be $\nu \to \mu$, i.e., $q \to 1$, so, in agreement with eq. (3), we do get the r-matrix. However, this would imply $\lim_{q\to 1} |\check{\beta}_l\rangle = |\alpha_l\rangle$ that is certainly wrong in general. The naive limit does not work since the L-operator itself varies as $\nu \to \mu$. In the correct limit, the time lattice spacing must approach zero independently of μ, ν.

Let us introduce v' – another "copy" of the discrete flow v with Miwa's variable ν', so now we have a 3D lattice. Equations of the type (13) are valid in the 2D sections $v' =$ const, $u =$ const, $v =$ const. Now we can tend $\nu' \to \mu$ leaving ν unchanged. Set $q' = \mu/\nu' = 1+\varepsilon+O(\varepsilon^2)$, $\varepsilon \to 0$, where ε is the lattice spacing in the direction $m' = 1/2(u-v')$. The discrete M-operators are defined up to multiplication by a scalar function of z independent of dynamical variables. It is convenient to normalize the M-operators by $\mathcal{M}_l(z) = I$ at $\varepsilon = 0$. Then the next term (of order ε) yields the continuous time M-operator. To find it, we expand in ε the discrete M-operator $\mathcal{M}_{B'_l \leftarrow C'_l}(z)$ which generates the shift $(l-1,l,1) \to (l,l,0)$ on the 3D lattice with coordinates (u,v,v').

Fig.2 displays the u,v'-section. Coordinates of the vertices are: $A'_l = (l-1,l,0)$, $B'_l = A_l = (l,l,0)$, $C'_l = (l-1,l,1)$, $D'_l = (l,l,1)$. The point C'_l tends to the point $B'_l = A_l$ as $\nu' \to \mu$, so the parallelogram collapses to the u-axis. We have: $\mathcal{M}_{B'_l \leftarrow C'_l}(z) = I + \varepsilon M_l(z) + O(\varepsilon^2)$, where

$$M_l(\mu z) = \frac{1}{z-z^{-1}} \begin{pmatrix} \frac{1}{2}(z+z^{-1})\frac{\tau(l-1,l)\hat{\tau}(l+1,l)}{\tau(l,l)\hat{\tau}(l,l)} & \frac{\tau(l-1,l)\tau(l+1,l)}{\tau(l,l)\hat{\tau}(l,l)} \\ \frac{\hat{\tau}(l-1,l)\hat{\tau}(l+1,l)}{\tau(l,l)\hat{\tau}(l,l)} & \frac{1}{2}(z+z^{-1})\frac{\hat{\tau}(l-1,l)\tau(l+1,l)}{\tau(l,l)\hat{\tau}(l,l)} \end{pmatrix}. \tag{21}$$

The r-matrix is $r(z) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\left[(z+z^{-1})^{-1}R^{(+)}(z;q') - I\otimes I\right]$, so

$$r(z) = \frac{1}{2(z-z^{-1})}\left[(z+z^{-1})I\otimes I + 2\sigma_1\otimes\sigma_1 + 2\sigma_2\otimes\sigma_2 + (z+z^{-1})\sigma_3\otimes\sigma_3\right]. \tag{22}$$

Comparing with (21), we get (3) with the r-matrix (22).

6. The main result of this work is the R-matrix representation (19) of the local L-M pair for the classical SG model in discrete space-time. In our opinion, the very fact that the typical quantum R-matrix naturally arises in a purely classical problem is important and interesting by itself. It would be desirable to clarify a connection with the quantum Yang–Baxter equation (which already arised in purely classical problems in a different context [19, 20]). We should stress that the "quantum" parameter q of the R-matrix in our context is related to the mass parameter and the lattice spacing of the classical model.

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