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PERTURBATION THEORY FOR THE $\pi^+\pi^-$ ATOM

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A perturbation-theory framework is developed for calculation of the characteristics of the $\pi^+\pi^-$ atom on the basis of the field-theoretical Bethe–Salpeter approach. A closed expression is obtained for the first-order correction to the $\pi^+\pi^-$ atom lifetime.

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Experimental studies of the hadronic atoms $\pi\pi$ [1], πp, and πd [2] have now been carried out. The first estimate of the $\pi^+\pi^-$ atom lifetime was given in Ref. [1]. The DIRAC collaboration is now designing an experiment at CERN for the high-precision measurement of the lifetime of $\pi^+\pi^-$ atoms. This experiment might provide a decisive improvement in the direct determination of the difference of the S-wave $\pi\pi$ scattering lengths and thus serve as a valuable test for the predictions of chiral perturbation theory [3]. In view of these experiments there arises a need for a theoretical framework which would enable one to calculate the characteristics of such atoms with a high accuracy on the basis of the ideas of standard model.

The theoretical study of hadronic atoms starts from Refs. [4–6], where the nonrelativistic formulas for the lifetime of a hadronic atom and the shift of its energy levels due to the strong interactions are obtained, which relate these quantities to the strong scattering lengths. The expression for the width Γ_0 of the $\pi^+\pi^-$ atom in the ground state is

$$\Gamma_0 = \frac{16\pi}{9}\sqrt{\frac{2\Delta m_\pi}{m_\pi}}\,(a_0^0 - a_0^2)^2\,\phi_0^2 \qquad (1)$$

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where Δm_π is the $m_{\pi^\pm} - m_{\pi^0}$ mass difference, and ϕ_0 is the value of the Coulomb wave function (w.f.) of pionium at the origin.

The approach to the problem of hadronic atoms which was developed in Ref. [4] makes use of a general characteristic feature of hadronic atoms — the factorization of strong and electromagnetic interactions. Formula (1) demonstrates this factorization property explicitly, expressing the atom lifetime as a product of two factors: the Coulomb w.f. at the origin, and the strong interaction factor, which is completely concentrated in the $\pi\pi$ strong scattering lengths.

The problem of evaluation of the electromagnetic and strong corrections to the basic formula (1) within different approaches is addressed in Refs. [7–14]. For a brief review see Ref. [12]. In that paper we derived the relativistic analog of formula (1) within the Bethe–Salpeter (BS) approach, taking into account the first-order correction due to the displacement of the bound state pole position by the strong interactions (strong correction). This correction was found to be of relative order 10^{-3}. It should be stressed that the field-theoretical approaches [10, 12–14] to the problem, unlike the potential treatment [7, 11], do not refer to the concept of a phenomenological strong interaction $\pi\pi$ potential, which is a source of additional ambiguity in calculations of the characteristics of hadronic atoms. In the former approaches these characteristics are expressed directly in terms of the underlying strong-interaction (chiral) Lagrangian, and the results can be compared to experiment, providing a consistent test of the predictions of the chiral theory.

In the present work we suggest a relativistic perturbation-theory framework for the calculation of the energy levels and lifetime of hadronic atoms. The main purpose of this work is to demonstrate the possibility (not only in potential scattering theory but in the BS treatment as well) of the clear-cut factorization of the strong and electromagnetic interactions in the observable characteristics of hadronic atoms, thus avoiding the double-counting problem in the calculation of these quantities. It should be noted that the suggested approach allows one to calculate the strong and electromagnetic corrections in all orders of perturbation theory. At the present stage we apply the general formalism to the calculation of the first-order strong and electromagnetic corrections to the pionium lifetime. The results for the strong corrections obtained in Ref. [12] are reproduced in these calculations.

Our approach is based on a perturbation expansion about the solution of the BS equation with a Coulomb kernel similar to that introduced in Ref. [15]:

$$V_C(\mathbf{p},\mathbf{q}) = \sqrt{w(\mathbf{p})}\,\frac{4im_\pi e^2}{(\mathbf{p}-\mathbf{q})^2}\,\sqrt{w(\mathbf{q})}, \qquad w(\mathbf{p}) = \sqrt{m_\pi^2 + \mathbf{p}^2} \tag{2}$$

The factor $\sqrt{w(\mathbf{p})w(\mathbf{q})}$ introduced in the kernel (2) enables one to reduce the BS equation with such a kernel to the exactly solvable Schrödinger equation with the Coulomb potential. Then the exact solution of the BS equation with this kernel is written in the form

$$\psi_C(p) = iG_0(M^*;p)\,4\sqrt{w(\mathbf{p})}\,\frac{4\pi\alpha m_\pi \phi_0}{\mathbf{p}^2+\gamma^2}, \quad \bar{\psi}_C(p) = \psi_C(p), \tag{3}$$

where $\gamma = m_\pi\alpha/2$, $M^{*2} = m_\pi^2(4-\alpha^2)$ is the eigenvalue corresponding to the unperturbed ground-state solution, and G_0 denotes the free Green's function of the $\pi^+\pi^-$ pair. The exact Green's function corresponding to the Coulomb kernel (2) is given by the well-known

expression

$$G_C(P^\star;p,q) = (2\pi)^4\delta^{(4)}(p-q)G_0(P^\star;p) + G_0(P^\star;p)T_C(E^\star;\mathbf{p},\mathbf{q})G_0(P^\star;q). \quad (4)$$

Here T_C is given by

$$T_C(E^\star;\mathbf{p},\mathbf{q}) = 16i\pi m_\pi\alpha\sqrt{w(\mathbf{p})w(\mathbf{q})}\left[\frac{1}{(\mathbf{p}-\mathbf{q})^2} + \int_0^1 \frac{\nu d\rho\rho^{-\nu}}{D(\rho;\mathbf{p},\mathbf{q})}\right] \quad (5)$$

$$D(\rho;\mathbf{p},\mathbf{q}) = (\mathbf{p}-\mathbf{q})^2\rho - \frac{m_\pi}{4E^\star}\left(E^\star - \frac{\mathbf{p}^2}{m_\pi}\right)\left(E^\star - \frac{\mathbf{q}^2}{m_\pi}\right)(1-\rho)^2,$$

where $\nu = \alpha\sqrt{m_\pi/(-4E^\star)}$ and $E^\star = (P^{\star 2} - 4m_\pi^2)/(4m_\pi)$.

The full BS equation for the $\pi^+\pi^-$ atom w.f. $\chi(p)$ is written as

$$G_0^{-1}(P;p)\chi(p) = \int \frac{d^4k}{(2\pi)^4}V(P;p,q)\chi(q), \quad (6)$$

where $V(P;p,q)$ denotes the full BS kernel, which is constructed from the underlying (effective) Lagrangian according to the general rules and includes all the strong and electromagnetic two-charged-pion irreducible diagrams. In particular, it contains the diagrams with two neutral pions in the intermediate state which govern the decay of the $\pi^+\pi^-$ atom into $\pi^0\pi^0$. Note that in addition $V(P;p,q)$ contains the two-particle reducible charged-pion self-energy diagrams attached to the outgoing pionic legs (with the relative momentum q). These diagrams arise in the definition of the kernel $V(P;p,q)$ because the free two-particle Green's function instead of the dressed one is used in the left-hand side of Eq. (6). The c.m. momentum squared P^2 of the atom has a complex value, corresponding to the fact that the atom is an unstable system. According to the conventional parametrization, we can write $P^2 = \bar{M}^2 = M^2 - iM\Gamma$ where M denotes the "mass" of the atom, and Γ is the atom decay width.

The full four-point Green's function $G(P)$ for the kernel V has a pole in the complex P^2 plane at the bound-state energy. The relation between the exact w.f. $\chi(p)$ and the Coulomb w.f. ψ_C is given by [12]

$$\langle\chi| = C\langle\psi_C|\,G_C^{-1}(P^\star)G(P), \quad P^{\star 2}\to M^{\star 2},\ P^2\to\bar{M}^2 \quad (7)$$

where C is the normalization constant. In what follows we assume that the limiting procedure is performed with the use of the prescription [12] $P^{\star 2} = M^{\star 2} + \lambda$, $P^2 = \bar{M}^2 + \lambda$, $\lambda\to 0$. The validity of Eq. (7) can be trivially checked by extracting the bound-state pole in $G(P)$ and using the BS equation for ψ_C.

In order to perform the perturbation expansion of the bound-state characteristics M and Γ about the unperturbed values, we, as in Ref. [12], split the full BS kernel V into two parts as $V = V_C + V'$ and consider V' as a perturbation. It can be shown that Eq. (7) is equivalent to

$$\langle\chi| = -C^{-1}\langle\psi_C|\left[1 + (\Delta G_0^{-1} - V')G_R Q\right]^{-1}, \qquad \Delta G_0^{-1} = G_0^{-1}(P) - G_0^{-1}(P^\star) \quad (8)$$

With the use of Eq. (8) the following identity is easily obtained

$$\langle\psi_C|\left[1 + (\Delta G_0^{-1} - V')G_R Q\right]^{-1}(\Delta G_0^{-1} - V')|\psi_C\rangle = 0, \quad (9)$$

which is an exact relation and serves as a basic equation for performing the perturbation expansion for the bound-state energy.

In Eqs. (8) and (9) G_RQ stands for the regular (pole-subtracted) part of the Coulomb Green's function (4), projected onto the subspace orthogonal to the ground-state unperturbed solution. This quantity can be further split into two pieces, according to $G_RQ = G_0(M^\star) + \delta G$. Here the function δG corresponds to the ladder of the exchanged Coulomb photons and thereby contains explicit powers of α. It is given by the following expression:

$$\begin{aligned}
\delta G &= i\sqrt{w(\mathbf{p})w(\mathbf{q})}\left[\Phi(\mathbf{p},\mathbf{q}) - S(\mathbf{p})S(\mathbf{q})\frac{8}{M^\star}\frac{\partial}{\partial M^\star}\right]G_0(M^\star,p)G_0(M^\star,q) \\
\Phi(\mathbf{p},\mathbf{q}) &= 16\pi m_\pi\alpha\left[\frac{1}{(\mathbf{p}-\mathbf{q})^2} + I_R(\mathbf{p},\mathbf{q})\right] + (m_\pi\alpha)^{-2}S(\mathbf{p})S(\mathbf{q})R(\mathbf{p},\mathbf{q}) \qquad (10) \\
S(\mathbf{p}) &= 4\pi m_\pi\alpha\phi_0(\mathbf{p}^2+\gamma^2)^{-1},\ R(\mathbf{p},\mathbf{q}) = 25 - \sqrt{\frac{8}{\pi m_\pi\alpha}}[S(\mathbf{p})+S(\mathbf{q})] + \cdots
\end{aligned}$$

where the ellipses stand for the higher-order terms in α. The integral $I_R(\mathbf{p},\mathbf{q})$ is given by

$$I_R(\mathbf{p},\mathbf{q}) = \int_0^1 \frac{d\rho}{\rho}\ [D^{-1}(\rho;\mathbf{p},\mathbf{q}) - D^{-1}(0;\mathbf{p},\mathbf{q})],\ E^\star = -\frac{1}{4}m_\pi\alpha^2 \qquad (11)$$

Equation (8) expresses the exact BS w.f. of the atom in terms of the unperturbed w.f. via the perturbation expansion in the perturbation potential V'. This potential consists of the following pieces.

1. The purely strong part, which is isotopically invariant. This part survives when the electromagnetic interactions are "turned off" in the Lagrangian.

2. A part containing the diagrams with finite mass insertions, which are responsible for the $m_{\pi^\pm} - m_{\pi^0}$ electromagnetic mass difference.

3. A part containing the exchanges of one, two, ... virtual photons and an arbitrary number of strong interaction vertices.

Note that the terms 1 and 2 are more important, for the following reasons. The first term includes the strong interactions, which are responsible for the decay of the pionium. The second term makes this decay kinematically allowed due to the finite difference of the charged and neutral pion masses. Consequently, it seems to be natural to consider pieces 1 and 2 together. We refer to the corresponding potential as V_{12}. The T matrix corresponding to the potential V_{12} is defined by $T_{12}(P) = V_{12}(P) + V_{12}(P)G_0(P)T_{12}(P)$. The rest of the potential V' is referred as $V_3 = V' - V_{12}$. In what follows we restrict ourselves to the first order in the fine structure constant α, i.e., we consider the diagrams with only one virtual photon contained in V_3.

Returning to the basic equation (9), we expand it in a perturbation series, treating V_3 and δG as perturbations. Meanwhile we expand ΔG_0^{-1} in a Taylor series in $\delta M = \bar{M} - M^\star$ and make the substitution $\bar{M} = M^\star + \Delta E^{(1)} + \Delta E^{(2)} - i/2\,\Gamma^{(1)} - i/2\,\Gamma^{(2)} + (8M^\star)^{-1}\Gamma^{(1)2} + \cdots$.

Restricting ourselves to the first order of the perturbation expansion, we arrive at the following relations:

$$\Delta E^{(1)} = \mathrm{Re}\left(\frac{i}{2M^\star}\frac{T_{12}}{m_\pi}\phi_0^2\right),\quad -\frac{1}{2}\Gamma^{(1)} = \mathrm{Im}\left(\frac{i}{2M^\star}\frac{T_{12}}{m_\pi}\phi_0^2\right). \qquad (12)$$

Hereafter we use the local approximation for T_{12}, assuming that it does not depend on the relative momenta. Equations (12) coincide with the well-known Deser-type formulas for the energy-level shift and lifetime [4]. Note that on the mass shell

$$\mathrm{Re}(iT_{12}) \sim T(\pi^+\pi^- \to \pi^+\pi^-), \quad \mathrm{Im}(iT_{12}) \sim \sqrt{\Delta m_\pi}|T(\pi^+\pi^- \to \pi^0\pi^0)|^2. \tag{13}$$

If we assume $V_3 = \delta G = 0$, we arrive at the result

$$\frac{\Gamma^{(2)}}{\Gamma^{(1)}} = -\frac{9}{8}\frac{\Delta E^{(1)}}{E_1} - 0.763\alpha, \quad \text{where} \quad E_1 = -\frac{1}{4}m_\pi\alpha^2. \tag{14}$$

The first term of this expression, called the "strong correction", was obtained in our previous paper [12]. However, as opposed to the present derivation, in Ref. [12] we used the Born approximation for the calculation of $\Delta E^{(1)}$, i.e., in Eq. (12) T_{12} was replaced by V_{12}. The second term comes from the relativistic normalization factor $\sqrt{w(\mathbf{p})w(\mathbf{q})}$ in the kernel (2) and corresponds to the relativistic modification of the pionium Coulomb w.f. $|\int d^4p/(2\pi)^4\psi_C(p)|^2 = \phi_0^2(1 - 0.381\alpha)^2/m_\pi$. Since this correction comes from the Coulomb w.f. of the atom, it does not depend on the parameters of the strong $\pi\pi$ interaction, and for this reason it was neglected in Ref. [12].

Inclusion of δG introduces a correction in the lifetime due to the exchange of an infinite number of Coulomb photons. The integrals emerging in the calculation of this correction are ultraviolet convergent, containing, however (in complete analogy with a well-known result from nonrelativistic scattering theory), an infrared enhancement $\alpha\ln\alpha$ which stems from the one-photon exchange piece in Eq. (5). Collecting all terms together and using Eqs. (12) for relating $\mathrm{Im}T_{12}$ to $\Delta E^{(1)}$, we finally arrive at the first-order correction to the pionium rate,

$$\Gamma = \Gamma_0\Big(1 + \underbrace{\Big(-\frac{9}{8}\frac{\Delta E^{(1)}}{E_1}\Big)}_{\text{strong}} + \underbrace{(-0.763\alpha)}_{\text{relativistic w.f.}} + \underbrace{(1/2 + 2.694 - \ln\alpha)\frac{\Delta E^{(1)}}{E_1}}_{\text{Coulomb photon exchanges}} +$$
$$+ \delta_M - \Big(M^*\Gamma^{(1)}\Big)^{-1}\mathrm{Re}\langle\psi_C|(1 + T_{12}G_0(M^*))V_3(1 + G_0(M^*)T_{12})|\psi_C\rangle\underline{igg}r\Big), \tag{15}$$

where δ_M stands for the mass shift correction [14] and the last term collects the radiative corrections [13, 14] (including the retardation correction [10], the correction due to vacuum polarization [8], etc.). In Eq. (15) all the first-order strong and electromagnetic corrections are given in closed form, thus avoiding any difficulties connected with the double-counting problem. The kernel which appears in the last term, $(1+T_{12}G_0(M^*))V_3(1+G_0(M^*)T_{12})$, is constructed from the underlying Lagrangian with the use of the conventional Feynman diagrammatic technique. A detailed reexamination of the above-mentioned corrections within the BS approach will be addressed in our forthcoming publications.

In order to estimate the size of the calculated corrections to the pionium lifetime (Eq. (15)) we have used the following value of the singlet scattering length: $m_\pi(2a_0^0 + a_0^2) = 0.49$, corresponding to a value $\Delta E^{(1)}/E_1 = 0.24\%$. The first, second, and third terms then contribute, respectively, -0.26%, -0.55%, and $+1.85\%$, and the total contribution amounts to $\sim$1% of the decay width (apart from the mass shift and radiative corrections). The largest contribution comes from the $\alpha\ln\alpha$ term in Eq. (15).

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