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SELF-SIMILAR POTENTIALS AND ISING MODELS

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A new link between soliton solutions of integrable nonlinear equations and one-dimensional Ising models is established. Translational invariance of the spin lattice associated with the KdV equation is related to self-similar potentials of the Schrödinger equation. This gives antiferromagnets with exponentially decaying interaction between the spins. Partition function is calculated exactly for a homogeneous magnetic field and two discrete values of the temperature.

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The one-dimensional Schrödinger equation

$$L\psi(x) \equiv -\psi_{xx}(x) + u(x)\psi(x) = \lambda\psi(x) \tag{1}$$

lies in the foundations of quantum mechanics and theory of solitons. The class of potentials $u(x)$, for which the spectrum and eigenfunctions of the operator L are known in the closed form, is of a particular interest. It includes simple potentials tied to the Gauss hypergeometric function (for a review, see [1]), finite-gap potentials, and the potentials whose discrete spectra consist of a number of arithmetic or geometric progressions (see [5, 3] and references therein). The latter potentials appear after a self-similar reduction of the factorization chain or the chain of Darboux transformations. In this note we discuss relation of the self-similar potentials to one-dimensional Ising type spin chain models. Below we use the language of the soliton theory described, e.g., in [4, 5].

It is well known that if the potential $u(x,t)$ and the wave function $\psi(x,t)$ in (1) depend on 'time' t in such a way that

$$\psi_t(x,t) = B\psi(x,t), \qquad B \equiv -4\partial_x^3 + 6u(x,t)\partial_x + 3u_x(x,t), \tag{2}$$

then the compatibility condition of (1) and (2), $L_t = [B, L]$, is equivalent to the Korteweg - de Vries (KdV) equation $u_t + u_{xxx} - 6uu_x = 0$. The N-soliton solution of this equation can be represented in the form $u(x,t) = -2\partial_x^2 \ln \tau_N(x,t)$, where $\tau_N = \det C$ is the determinant of the matrix

$$C_{ij} = \delta_{ij} + \frac{2\sqrt{k_i k_j}}{k_i + k_j} e^{(\theta_i + \theta_j)/2}, \qquad \theta_i = k_i x - k_i^3 t + \theta_i^{(0)}. \tag{3}$$

Here k_i are the amplitudes of solitons related to the bound state energies of (1), $\lambda_i = -k_i^2/4$, and $\theta_i^{(0)}/k_i$ are the zero time phases. The ordering $0 < k_N < \ldots < k_1$ is assumed.

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Equivalently, this τ-function can be rewritten in the form [4, 5]:

$$\tau_N = \sum_{\mu_i=0,1} \exp\Big(\sum_{1\le i<j\le N} A_{ij}\mu_i\mu_j + \sum_{1\le i\le N} \theta_i\mu_i \Big), \tag{4}$$

where the phase shifts A_{ij} are determined by the formula

$$e^{A_{ij}} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}. \tag{5}$$

There are generalizations of the expressions (3)-(5) such that the corresponding $u(x, t, \ldots)$ satisfy higher order members of the KdV-hierarchy, sin-Gordon, Kadomtsev – Petviashvili (KP), Toda, and some other integrable equations [4].

We start from the observation that the expression (4) has nice interpretation within the statistical mechanics. Namely, for $\theta_i = \theta^{(0)} =$ const it defines the grand partition function of the lattice gas model [6]. In this case μ_i play the role of filling factors of the lattice sites by repulsing molecules, $\theta^{(0)}$ is proportional to the chemical potential, and A_{ij} are proportional to the interaction energy between the i-th and j-th molecules.

Simultaneously, (4) is closely related to the partition function of the one-dimensional Ising model [6]:

$$Z_N = \sum_{\sigma_i=\pm 1} e^{-\beta E}, \qquad E = \sum_{i<j} J_{ij}\sigma_i\sigma_j - \sum_{1\le i\le N} H_i\sigma_i, \tag{6}$$

where N is the number of spins $\sigma_i = \pm 1$, $J_{ij} = J_{ji}$ is the coupling between i-th and j-th spins, H_i is the external magnetic field, $\beta = 1/kT$ is the inverse temperature. Indeed, let us introduce into (4) the spin variables via the substitution $\mu_i = (\sigma_i + 1)/2$. After some simple calculations one finds

$$\tau_N = e^{\Phi} Z_N, \qquad \Phi = \tfrac{1}{4}\sum_{i<j} A_{ij} + \tfrac{1}{2}\sum_{1\le j\le N} \theta_j, \tag{7}$$

provided

$$A_{ij} = -4\beta J_{ij}, \qquad \theta_i = 2\beta\Big(H_i + \sum_{1\le j\ne i\le N} J_{ij}\Big). \tag{8}$$

As a result, one arrives at an interesting fact: from a given N-soliton τ-function of the KdV equation (4), one recovers the partition function of the N-spin Ising model (7). The τ-function is defined only up to a gauge factor $\exp(ax + b)$, and (7) fits this freedom. Therefore one may identify the N-soliton τ-function itself with (6) for the specific exchange interaction (5). This fact alone does not help much in the evaluation of Z_N. However, the recursive way of building N-soliton potentials with the help of Darboux transformations or the factorization method appears to be quite useful. Let us provide the representation of Z_N following from the Wronskian form of τ_N [7, 8]

$$Z_N = \frac{2^{N(N+1)/2}\, W_N}{\prod_{i<j}(k_i^2 - k_j^2)^{1/2}}, \qquad W_N = \det\left(\frac{d^{i-1}\Psi_j}{dx^{i-1}}\right), \tag{9}$$

where $\Psi_{2j-1} = \mathrm{ch}\,\beta H_{N-2j+2}, \Psi_{2j} = \mathrm{sh}\,\beta H_{N-2j+1}$. Dependence of H_j on the soliton parameters is read from (8).

The factorization method transforms a given potential $u_j(x) = f_j^2(x) - f_{jx}(x) + \lambda_j$ with some discrete spectrum to the potential $u_{j+1}(x) = u_j(x) + 2f_{jx}(x)$ containing an additional (the lowest) bound state with the prescribed energy λ_j. Within the Ising models context, this corresponds to the extension of the lattice by one more site. Then the infinite-soliton potentials correspond to the thermodynamic limit $N \to \infty$. Characterization of general τ_N at $N \to \infty$ is a challenging problem, but for the specific choice of parameters $k_i, \theta_i^{(0)}$ this function can be analyzed to some extent through the basic infinite chain of equations [1]

$$(f_j(x) + f_{j+1}(x))_x + f_j^2(x) - f_{j+1}^2(x) = \rho_j \equiv \lambda_{j+1} - \lambda_j, \quad j \in \mathbf{Z}. \tag{10}$$

In general both τ_N and Z_N diverge in the limit $N \to \infty$. If the corresponding solutions of (10) are finite, then the divergences gather into the gauge factor.

A key observation of the present work is that the simplest physical constraints imposed upon the form of spin interactions J_{ij} of the infinite Ising chain select the potentials with the discrete spectrum composed from a number of geometric progressions. First, let us demand that all the spins are situated on equal distance from each other and that they are identical, i.e. that there is a translational invariance, $A_{i+1,j+1} = A_{ij}$. This means that the intensities of interaction A_{ij} depend only on the distance between the sites $|i-j|$, $A_{ij} = A(|i-j|)$. Such a natural constraint has the *unique* solution

$$k_i = k_1 q^{i-1}, \qquad q = e^{-2\alpha}, \qquad A_{ij} = 2\ln|\tanh\alpha(i-j)|, \tag{11}$$

where $\alpha > 0$ is an arbitrary constant. For finite N this spectrum corresponds to reflectionless potentials with the eigenvalues condensing near $\lambda = 0$. For $q > 1$, one should write $k_i = k_1 q^{-i+1}$ for correct ordering of k_i. (The exponentially growing spectrum is formally obtained for purely imaginary k_1 and $q > 1$, but the corresponding potential contains singularities.) In the $N \to \infty$ limit, one gets an infinite soliton potential with the discrete spectrum $\lambda_j = -k_1^2 q^{2(j-1)}/4$ describing a specific semi-infinite spin chain (j takes only positive values). As $q^j \to 0$ for $j \to \infty$, the x and t depending part of the magnetic field is decaying exponentially from the edge of the lattice. The limits $x, t \to \infty$ correspond to the growing depth of penetration of the magnetic field inside the bulk. Note that one can analyze boundary effects by working with a difference of the free energy at two fixed values of the magnetic field.

Since $0 < |\tanh\alpha(i-j)| < 1$, one has $J_{ij} > 0$, i.e. an antiferromagnetic interaction (the spins are not aligned in the ground state). It has nice physical characteristics – its intensity falls exponentially fast with the distance between the sites. It is well known that the one-dimensional systems with finite range interactions do not have phase transitions at non-zero temperature. There is a model with the exponential interaction $J_{ij} = -\gamma|J_0|e^{-\gamma|i-j|}$ solved in the limit $\gamma \to 0$ by M. Kac [9]. This limit corresponds to the very weak but long-range interaction and shows a phase transition with the Van der Waals equation of state.

There should be some relation of our model to the Kac one, but it is not clear whether there exists a direct connection. A similar molecular approximation limit is reached in our case if $\alpha \to 0$. Formally $A_{ij} \propto J_{ij}/kT$ diverge in this limit. If we renormalize interaction constants $J_{ij}^{ren} = J_{ij}(q^{-1} - q)$ and the temperature $T_{ren} = T(q^{-1} - q)$, then the maximal interaction energy of a single i-th spin (determined by the summation of J_{ij}^{ren} over j) will be finite for $\alpha \to 0$ (or $q \to 1$). Therefore the limit $q \to 1$ corresponds to the *long range interaction* model *at low temperature.* Note that one should rescale simultaneously the magnetic field $H = h/(q^{-1} - q)$ to imitate the change of the temperature.

The particular form of the renormalization factor $q^{-1} - q$ was chosen in order in the limit $q \to 0$ to recover the interaction $J_{ij}^{ren} \propto \delta_{i+1,j}$. If one takes h as a real magnetic field then one gets the nearest neighbor interaction Ising model at high temperature. If the magnetic field is not rescaled then the $q \to 0$ limit corresponds to the completely non-interacting spins. Thus our formalism allows to analyze partition function upon two dimensional planes in the space of variables (T, H, q). Unfortunately, for fixed q the temperature is fixed as well and we may normalize the "KdV temperature" to $kT = 1$.

The discrete spectrum does not characterize completely even the reflectionless potentials — one has to fix the phases θ_i. Only for the special choice of these parameters one arrives at the self-similar potentials. E.g., the simplest case is determined by the condition that the scaling of x and t by q and q^3 respectively is equivalent to removing of one soliton. Formally this corresponds to the constraint $\theta_i(qx, q^3 t) = \theta_{i+1}(x, t)$ assuming the choice $\theta_i^{(0)} = \theta^{(0)} =$ const. However, τ_N, Z_N and Φ in (7) are diverging for $N \to \infty$ and a more careful analysis is thus called for. Note that the shift of H_i in (8) remains finite and it becomes a fixed constant for $i \to \infty$. This means that in the thermodynamic limit the zero chemical potential in the lattice gas partition function corresponds to a fixed nonzero magnetic field in the Ising model, and, vice versa, zero magnetic field matches with a prescribed value of the chemical potential.

Let us consider now the "M-color" Ising model for which the chain is formed by the embedded sublattices when the blocks of M spins with different distances between them are periodically repeated. Within each of this block the distances between spins are not equal, so that the interaction constants between the first M sites are given by arbitrary (random) numbers. Equivalently, one may think that in the equal distance lattice points one has different magnetic moment particles, i.e. some kind of ferrimagnetic interaction. Such physical constraints are bound to the condition $A_{i+M,j+M} = A_{ij}$, which leads to the constraint upon the soliton energies of the form $k_{j+M} = qk_j$, generalizing the previous case. For a specific choice of the phases $\theta_{i+M}^{(0)} = \theta_i^{(0)}$ one arrives at the general self-similar potentials for which one has $\theta_i(qx, q^3 t) = \theta_{i+M}(x, t)$. The rigorous definition of these potentials for fixed time is given by the constraints [9]

$$f_{j+M}(x) = q f_j(qx), \qquad \rho_{j+M} = q^2 \rho_j \tag{12}$$

imposed upon the chain (10). The system of mixed differential and q-difference equations arising after this reduction describes q-deformation of the Painlevé transcendents and their higher order analogs. For $M = 1$ one has a q-harmonic oscillator model, for $M = 2$ a system with the $su_q(1,1)$ symmetry algebra, etc.

Using the Wronskian representation (9), we calculated exactly the free energy per site f_I in the thermodynamic limit $Z_N \to e^{-\beta N f_I}, N \to \infty$, for a homogeneous magnetic field and arbitrary M. For $M = 1$ one has

$$-\beta f_I(H) = \ln \frac{2(q^4; q^4)_\infty \operatorname{ch} \beta H}{(q^2; q^2)_\infty^{1/2}} + \frac{1}{2\pi} \int_0^\pi d\nu \ln \left(|\rho(\nu)|^2 - q \tanh^2 \beta H \right), \tag{13}$$

where $(a; q)_\infty = \prod_{j=0}^\infty (1 - aq^j)$ and

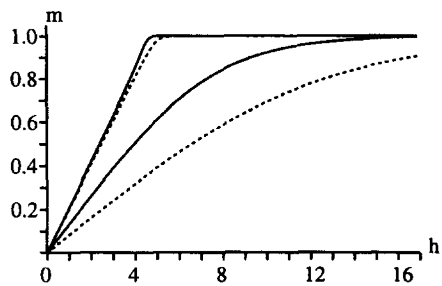
$$|\rho(\nu)|^2 = \frac{(q^2 e^{2i\nu}; q^4)_\infty^2 (q^2 e^{-2i\nu}; q^4)_\infty^2}{4 \sin^2 \nu \, (q^4 e^{2i\nu}; q^4)_\infty^2 (q^4 e^{-2i\nu}; q^4)_\infty^2} = q \frac{\theta_4^2(\nu, q^2)}{\theta_1^2(\nu, q^2)}. \tag{14}$$

The Jacobi θ-functions are defined in the standard way [10]. The density function $\rho(\nu)$ has integrable singularities near the $\nu = 0, \pi$ points. Note that it satisfies a curious identity such that the second term in (13) vanishes for $H = 0$.

Dependence of the magnetization $m(H) = -\partial_H f_I(H)$ on H looks as follows

$$m(H) = \left(1 - \frac{1}{\pi}\int_0^\pi \frac{\theta_1^2(\nu,q^2)d\nu}{\theta_4^2(\nu,q^2)\,\mathrm{ch}^2\,\beta H - \theta_1^2(\nu,q^2)\,\mathrm{sh}^2\,\beta H}\right)\tanh\beta H. \tag{15}$$

We substitute into this expression $\beta H = h/(q^{-1} - q)$ and plot $m(h)$ in Figure by the dashed lines for $q = 0.1$ (the lower curve) and $q = 0.8$. We would like to note that it is not clear how to solve the considered Ising model with the help of the traditional Bethe ansatz and transfer matrix methods [6].



Dependence of the magnetization $m(h)$ on h for the KdV case $n = 1$ (dashed lines) and for the BKP case $n = 2$ (solid lines). The lower curves correspond to $q = 0.1$ and the upper ones to $q = 0.8$

As was mentioned, a drawback of the given construction is that the KdV-generated partition function has a fixed temperature for fixed α. In order to obtain the full thermodynamical description it is necessary to extend the formalism and replace A_{ij} (11) at least by nA_{ij}, where n is a positive integer. The KdV temperature is thus normalized to $\beta = n = 1$ (for $n > 1$ one has to renormalize the magnetic field $H_i \to nH_i$ in order to imitate the effect of the temperature lowering). This means that we need to look for an integrable model with the phase shifts given by the powers of (5). Then one may hope to recover the partition function with arbitrary values of the temperature $\propto 1/n$ by an analytic continuation.

The phase shifts A_{ij} for a given Hirota polynomial $P(x_1, x_3, \ldots)$, determining a particular evolution equation, can be represented in the form [5]

$$e^{A_{ij}} = -\frac{P(k_1 - k_2, k_2^3 - k_1^3, \ldots, (-1)^\ell(k_1^{2\ell+1} - k_2^{2\ell+1}))}{P(k_1 + k_2, -k_2^3 - k_1^3, \ldots, (-1)^\ell(k_1^{2\ell+1} + k_2^{2\ell+1}))}, \tag{16}$$

where ℓ is the number of variables in P. We have looked for equations admitting N-soliton solutions with the prescribed phase shifts, substituting homogeneous (with the account of weights of the variables) polynomials with undefined coefficients into (16). It turns out that the taken conditions are very restrictive. The only solution we were able to find is the hierarchy which starts from $P(x_1, x_3, x_5) = 16x_1^6 + 20x_1^3x_3 + 9x_1x_5 - 5x_3^2$ corresponding to $n = 2$. After an appropriate rescaling of variables this polynomial coincides with the one for B-type KP (BKP) equation [11].

Using the Pfaffian representation of the N-soliton solutions of the BKP equation [11], we calculated exactly the partition function in the thermodynamical limit $N \to \infty$ for a homogeneous magnetic field and arbitrary M. For $M = 1$ one has

$$-\beta f_I(H) = \frac{1}{2\pi}\int_0^\pi d\nu \ln 2\left|\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\,\mathrm{ch}\,4\beta H + \frac{\partial_\nu\theta_1(\nu,q^{1/2})}{\theta_2(\nu,q^{1/2})}\right|, \tag{17}$$

where ∂_ν means the derivative with respect to the variable ν and $\theta_2(\nu, q^{1/2})$ is another Jacobi θ-function [10]. The dependence of magnetization on H is

$$m(H) = \left(1 - \frac{1}{\pi}\int_0^\pi d\nu \left(1 + \frac{(q;q)^2_\infty \theta_2(\nu, q^{1/2})\,\mathrm{ch}\,4\beta H}{(-q;q)^2_\infty \partial_\nu \theta_1(\nu, q^{1/2})}\right)^{-1}\right) \tanh 4\beta H. \qquad (18)$$

For $q \to 0$ one gets the simple answer $m(H) = \tanh 2\beta H$.

We substitute into (18) $\beta H = h/(q^{-1} - q)$ and plot $m(h)$ in the Figure by the solid lines for $q = 0.1$ (the lower curve) and $q = 0.8$. From the comparison of the magnetization curves one can see that with the lowering of temperature, which corresponds both to the transition from $n = 1$ to $n = 2$ and to the growing of q, $m(h)$ becomes more steep. This may be interpreted as a trend towards formation of a staircase-like fractal function that should take place at zero temperature according to the arguments of [12]. Formation of the platos for $m(h)$ at low temperatures can be easily checked analytically for the nearest neighbor interaction Ising antiferromagnet.

The attempts to find integrable systems with $n > 2$ have failed for Hirota polynomials of up to 20-th degree. Probably one has to pass from the scalar Lax pairs to the matrix ones in order to imitate other values of the discrete temperature. The lattice of temperatures itself resembles a discrete variable unifying different hierarchies of integrable systems into one class.

A relation between the two-dimensional nearest neighbor interaction Ising model and the sinh-Gordon hierarchy was discussed in [13]. In particular, the corresponding N-soliton solution τ-function, $N \to \infty$, was shown to be the generating function of correlation functions. It should be noted that our identification of the one-dimensional Ising model partition function with τ-functions of some integrable equations is different from the constructions considered in [13] and earlier related works. However, it is expected that the self-similar potentials (or q-analogs of the Painlevé transcendents) are related to some correlation functions in the corresponding setting as well. A hint on this comes from the fact that the supersymmetric quantum mechanical representation of the factorization method is related to the Lax pair of the sinh-Gordon equation.

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