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Low-energy gluon contributions to the vacuum polarization of heavy quarks

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We calculate a correction to the effective electromagnetic current at low energies induced by a heavy quark loop and determine the analytic structure of the vacuum polarization function at small q^2 for which an explicit expression at the $O(\alpha_s^3)$ order of perturbation theory is given. Implications to the high precision analysis of experimental data on heavy quark production in e^+e^--annihilation are discussed.

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High precision tests of the standard model remain one of the main topics of particle phenomenology. The recent observation of a possible signal from the Higgs boson may complete the experimentally confirmed list of the standard model particles [1]. With improving precision of experimental data more accurate theoretical formulae will be necessary for extracting numerical values of the standard model parameters. Recently an essential progress in high-order perturbation theory calculations for heavy quarks has been made where a number of new physical effects have been described theoretically with high accuracy. The cross section of top–antitop production near the threshold has been calculated at next-to-next-to-leading order of an expansion in the strong coupling constant and the top quark velocity with an exact account for Coulomb interaction (as a review, see Ref. [2]). This allows for the precision determination of a numerical value of the top quark mass at the next linear collider. The Coulomb resummation resides on a nonrelativistic approximation for the quark-antiquark system near the threshold that has been successfully used for the description of heavy quark properties within the operator product expansion techniques and sum rules [3 – 5]. Being applied to the $b\bar{b}$ system this method gives the best estimates of the b quark mass parameters [6 – 9].

In the present note we discuss a contribution of massless intermediate states to the correlators of heavy quark currents. For the correlator of the vector currents such a contribution first appears at the $O(\alpha_s^3)$ order of perturbation theory and is given by a three-gluon state. This gluon contribution to the correlator has a qualitatively new feature – its absorptive part starts at zero energy in contrast to other contributions where the absorptive parts start at the two-particle threshold. This feature determines the analytic structure of the correlator at small q^2 – at the $O(\alpha_s^3)$ order of perturbation theory a cut along the positive semiaxis emerges. The non-analyticity at the origin resulting from such a cut leads to strong limitations on the observables that can be theoretically constructed for confronting to experimental data. Because the data are most precise near the production threshold the theoretical analysis should enhance this part of the spectrum. Technically an enhancement of the near-threshold contributions is achieved by considering integrals of the production rate with weight functions which suppress the high-energy tail of the spectrum. The integrals with weight functions $1/s^n$ for different positive integer n, $s = E^2$, where E is the total center-of-mass energy of the produced particles, are called moments of the spectral density and most often used in the sum rule analysis. Theoretically such moments are given by the derivatives at $q^2 = 0$ of the vacuum polarization function $\Pi(q^2)$ which is a basic quantity for the analysis of the heavy quark production in the $J^{PC} = 1^{--}$ channel. The vacuum polarization function is given by

$$i\int\langle Tj_\mu(x)j_\nu(0)\rangle e^{iqx}d^4x = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2) \quad (1)$$

with the vector current $j^\mu = \bar\psi\gamma^\mu\psi$ of a heavy fermion ψ of the mass m. With the spectral density $\rho(s)$ defined by the relation

$$\rho(s) = \frac{1}{2\pi i}(\Pi(s+i0) - \Pi(s-i0)), \quad s > 0, \quad (2)$$

the dispersion representation

$$\Pi(q^2) = \int\frac{\rho(s)ds}{s-q^2} \quad (3)$$

holds. The integral in Eq. (3) runs over the whole spectrum of the correlator in Eq. (1) or over the whole support of the spectral density $\rho(s)$ in Eq. (2). A necessary ultraviolet regularization (subtractions, for instance) is assumed in Eq. (3). The moments of the spectral density $\rho(s)$ of the form

$$\mathcal{M}_n = \int \frac{\rho(s)ds}{s^{n+1}} \tag{4}$$

are usually studied within the sum rule method for heavy quarks [10]. These moments are related to the derivatives of the vacuum polarization function $\Pi(q^2)$ at the origin,

$$\mathcal{M}_n = \frac{1}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=0}. \tag{5}$$

Such moments are chosen in order to suppress a high energy part of the spectral density $\rho(s)$ which is not measured accurately in experiments. Within the sum rule method one believes that the theoretical expressions for the moments in Eq. (4), or, equivalently, the derivatives in Eq. (5) exist, i.e. formally lead to well-defined quantities, for any n. The existence of moments seems to be obvious because of the implicit assumption that the spectral density $\rho(s)$ of the correlator of the heavy quark electromagnetic currents vanishes below the two-particle threshold $s = 4m^2$, which means that the vacuum polarization function of heavy quarks $\Pi(q^2)$ is analytic in the whole complex plane of q^2 except for the cut along the positive real axis starting from $4m^2$. This assumption about the analytic properties of the vacuum polarization function $\Pi(q^2)$ is valid at the first few orders of perturbation theory. However, its validity in a full theory depends of the details of the interaction. For instance, the resummation of Coulomb effects to all orders may result in the appearance of bound states below the perturbation theory threshold $s = 4m^2$. Still this is only true for the attractive Coulomb interaction while the repulsive Coulomb interaction modifies the shape of the free particle spectrum but does not change its support, i.e. does not lead to bound state formation below the continuum spectrum. The assumption that the moments in Eq. (4) exist for any n may also be wrong at high orders of perturbation theory in models with massless particles, for example, in QCD with massless gluons. In QCD, at the $O(\alpha_s^3)$ order of perturbation theory there is a contribution of massless states to the correlator in Eq. (1) that leads to the infrared (small s) divergence of theoretical expressions for the moments for large n because of the branching point (cut) singularity of $\Pi(q^2)$ at the origin. We determine the behaviour of the vacuum polarization function $\Pi(q^2)$ at small q^2 ($q^2 \ll m^2$) as

$$\Pi(q^2)|_{q^2\approx 0} = \frac{C_g}{12\pi^2}\left(\frac{q^2}{4m^2}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right), \tag{6}$$

with

$$C_g = \frac{17}{243000} d_{abc}d_{abc}\left(\frac{\alpha_s}{\pi}\right)^3. \tag{7}$$

Here d_{abc} are the totally symmetric structure constants of the $SU(N_c)$ gauge group defined by the relation $d_{abc} = 2\mathrm{tr}(\{t^a, t^b\}t^c)$, and t^a are generators of the group with normalization $\mathrm{tr}(t^a t^b) = 1/2$. For the $SU(3)$ gauge group of QCD one has $d_{abc}d_{abc} = 40/3$. The parameter μ in Eq. (6) is the renormalization point.

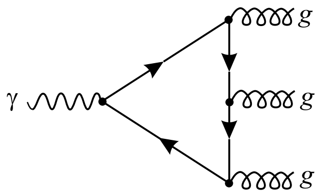
The singularity of the vacuum polarization function given in Eq. (6) (a cut along the positive real axis in a complex q^2-plane) prevents one from calculating moments of the spectral density in Eq. (4) with $n \geq 4$. Indeed, the high order derivatives of $\Pi(q^2)$ at the origin determining the high order moments according to Eq. (5) do not exist for $n \geq 4$ because of a branching point singularity as one can see from Eq. (6). In terms of the moments one can see this by calculating the behaviour of the spectral density at small squared energies s,

$$\rho(s)|_{s\approx 0} = \frac{C_g}{12\pi^2}\left(\frac{s}{4m^2}\right)^4, \tag{8}$$

which makes the integrals in Eq. (4) divergent at small s for $n \geq 4$. The formulae for the vacuum polarization function in Eqs. (6) and (7) are given for a heavy quark in the $SU(N_c)\otimes U(1)$ gauge model. The result for QED may be obtained with substituting $\alpha_s \to \alpha$ for the coupling constant and by changing the group factors; the contribution is, however, very small and of no practical interest. Contributions of light (massless) quarks appear at the $O(\alpha_s^4)$ order of perturbation theory and are neglected.

We present a derivation of our result given in Eqs. (6) and (7) and briefly discuss some consequences for the phenomenology of heavy quarks. Note that the induced current is a correction of order $1/m^4$ in the inverse heavy quark mass which vanishes in the limit of an infinitely heavy quark. Corrections in inverse heavy quark masses are important for tests of the standard model at the present level of precision and have been already discussed in various areas of particle phenomenology [11–13].

A correction to the electromagnetic current due to a virtual heavy quark loop is given by a transition of the photon to three gluons (see Figure). Two-gluon transitions are forbidden according to a generalization of Farry's theorem to nonabelian theories [14]. We are interested in the behaviour of the transition amplitude at low energies and take the limit of a very heavy quark. Formally the limit $m \to \infty$ is taken which in physical



Heavy quark loop correction to the electromagnetic current

terms means that m is much larger than all momenta of external legs of the diagram, namely the three gluons and the photon. The induced current J^μ is written in a covariant form as a derivative of an antisymmetric operator $\mathcal{O}_{\mu\nu}$ built from the gluon fields only,

$$J^\mu = \partial_\nu \mathcal{O}^{\mu\nu}, \quad \mathcal{O}^{\mu\nu} + \mathcal{O}^{\nu\mu} = 0. \tag{9}$$

This structure of the induced current automatically guarantees the current conservation

$$\partial_\mu J^\mu = 0 \tag{10}$$

as it should be for the electromagnetic current. A straightforward calculation gives the result for the induced correction

$$J^\mu = \frac{-g_s^3}{1440\pi^2 m^4}\left(5\partial_\nu \mathcal{O}_1^{\mu\nu} + 14\partial_\nu \mathcal{O}_2^{\mu\nu}\right), \tag{11}$$

with

$$\mathcal{O}_{1\mu\nu} = d_{abc} G^a_{\mu\nu} G^b_{\alpha\beta} G^c_{\alpha\beta}, \; \mathcal{O}_{2\mu\nu} = d_{abc} G^a_{\mu\alpha} G^b_{\alpha\beta} G^c_{\beta\nu} \tag{12}$$

where $G^a_{\mu\nu}$ is a gauge field strength tensor for the gauge group $SU(N_c)$.

A correlator of the induced current J^μ has the general form

$$\langle TJ^\mu(x)J^\nu(0)\rangle = -\partial_\alpha\partial_\beta \langle T\mathcal{O}^{\mu\alpha}(x)\mathcal{O}^{\nu\beta}(0)\rangle \tag{13}$$

where an explicit expression of the current as a derivative of the antisymmetric operator $\mathcal{O}^{\mu\nu}$ has been employed. The resulting correlator $\langle T\mathcal{O}^{\mu\alpha}(x)\mathcal{O}^{\nu\beta}(0)\rangle$ in Eq. (13) contains only gluonic operators. At leading order the correlator in Eq. (13) has a topological structure of a sunset diagram which is readily computed in configuration space [15]. We find

$$\langle TJ_\mu(x)J_\nu(0)\rangle =$$
$$= -\frac{34}{2025\pi^4 m^8}\left(\frac{\alpha_s}{\pi}\right)^3 d_{abc}d_{abc}\left(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2\right)\frac{1}{x^{12}}. \tag{14}$$

The Fourier transform of the correlator in Eq. (14) gives

$$i\int \langle TJ_\mu(x)J_\nu(0)\rangle e^{iqx} d^4x =$$
$$= (q_\mu q_\nu - g_{\mu\nu}q^2)\frac{C_g}{12\pi^2}\left(\frac{q^2}{4m^2}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right), \tag{15}$$

with the constant C_g taken from Eq. (7). The spectral density of the polarization function in Eq. (15) is given in Eq. (8).

The spectral density of the correlator in Eq. (14) can be found without an explicit calculation of its Fourier transform. Instead one can use a spectral decomposition (dispersion representation) in configuration space,

$$\frac{i}{x^{12}} = \frac{\pi^2}{2^8\Gamma(6)\Gamma(5)}\int_0^\infty s^4 D(x^2, s)ds, \tag{16}$$

with $D(x^2, s)$ being the propagator of a scalar particle of mass $\sqrt{s}$,

$$D(x^2, m^2) = \frac{im\sqrt{-x^2}K_1(m\sqrt{-x^2})}{4\pi^2(-x^2)}, \tag{17}$$

where $K_1(z)$ is a McDonald function (a modified Bessel function of the third kind, see e.g. Ref. [16]). $\Gamma(z)$ is Euler's gamma function.

Note that the three-gluon contribution to the spectral density of the quark current correlator in Eq. (1) for large energies (when the limit of massless quarks can be used) is well-known and reads [17 – 19]

$$\left(\frac{\alpha_s}{\pi}\right)^3 \frac{d_{abc}d_{abc}}{1024}\left(\frac{176}{3} - 128\zeta(3)\right). \tag{18}$$

Here $\zeta(z)$ is Riemann's ζ function.

One immediate application of our result is the precision determination of heavy quark parameters from the data on heavy quark production. Because of the low-energy gluon contributions, the large n ($n \geq 4$) moments of the spectral density do not formally exist. The range of n used in original considerations of sum rules and in some recent analyses requires a proper modification of theoretical expressions for the moments in order to account for this new contribution at the $O(\alpha_s^3)$ level of precision. For the $c\bar{c}$ system the moments with $n \sim 3 \div 7$ [10, 20] were analyzed. For the precision analysis of $b\bar{b}$ production the sum rules for the moments with larger $n \sim 5 \div 20$ were studied in the literature. In view of our result the modification of the analysis is necessary at the formal level of the $O(\alpha_s^3)$ accuracy of perturbation theory. Note that the considered correction is of the same order in α_s as the contribution of Coulomb bound states that is known to be numerically important for the description of Υ resonances. The restriction to use only first few moments with $n < 4$ seems not to be satisfactory from the phenomenological point of view. For small n the high-energy contribution which is not known experimentally with a reasonable precision is not sufficiently suppressed and introduces a large quantitative uncertainty into sum rules for the moments.

A modified analysis can be based on the theoretical expression for the derivatives of the correlator at some infrared safe point $q^2 = -\Delta < 0$ [21]. For the infrared regularized moments

$$\mathcal{M}_n(\Delta) = \frac{1}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=-\Delta} = \int_0^\infty \frac{\rho(s)ds}{(s+\Delta)^{n+1}} \tag{19}$$

there is no divergence at small s. The infrared regularization parameter Δ should be small because the continuum contribution to moments is not suppressed for large values of Δ even for sufficiently large n. However, it cannot be arbitrary small because the resulting correlator of gluonic currents at low energies in Eq. (13) is essentially normalized at $\mu^2 = \Delta$ when radiative corrections are taken into account. Therefore, Δ should be large enough for perturbation theory calculations to be justified [22, 23]. On the experimental side of the sum rules it is rather difficult to evaluate the part of the spectral integral over the low-energy gluons because of mixing with light quark contributions. These theoretical and experimental constraints on the numerical value of the parameter Δ require a special analysis of the accuracy attainable with the infrared regularized sum rules in Eq. (19).

Another possibility to bypass the problem of infrared divergence is to use finite energy sum rules without $1/s^n$ weight functions that are free from the infrared problem [24], or to apply a direct subtraction of the three-gluon contribution as has been recently proposed [25].

To conclude, we have presented a correction to the electromagnetic current induced by a virtual heavy quark loop and relevant for an effective theory of light degrees of freedom at low energies. The spectrum of the correlator of such an induced current starts at zero energy. This fact necessitates a modification of the standard analysis of the moment sum rules for the $b\bar{b}$ system at the $O(\alpha_s^3)$ order of perturbation theory.

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