Shedding and interaction of solitons in imperfect medium

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The propagation of a soliton pattern through one-dimensional medium with weakly disordered dispersion is considered. Solitons, perturbed by this disorder, radiate. The emergence of a long-range interaction between the solitons, mediated by the radiation, is reported. Basic soliton patterns are analyzed. The interaction is triple and is extremely sensitive to the phase mismatch and relative spatial separations within the pattern. The phenomenon is a generic feature of any problem explaining adiabatic evolution of solitons through a medium with frozen disorder.

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We consider long-range soliton interaction mediated by radiation in nonlinear $1d$ system with frozen disorder. The problem is of a great importance for nonlinear fiber optics of next generation (see e.g. [1, 2]) and it is also of general relevance for any of the traditional fields, like plasma physics, where propagation of solitary waves is possible. Our aim here is to answer the following sets of fundamental questions:

What statistics does describe the radiation emitted due to disorder by a single soliton, a pattern of solitons? How far do the radiation wings extend from the peak of the soliton(s)? What is the structure of the wings?

How strong is the radiation mediating interaction between the solitons? How is the interaction modified if we vary the soliton positions and phases within a pattern of solitons?

We focus on the dynamics of wave packets. The universal coarse-grained description of a wave packet envelope is given by the Nonlinear Schrödinger equation (NLS) [3 – 5]. We consider the $1d$ problem motivated mainly by applications to fiber optics [6]

$$i\partial_z\psi + d(z)\partial_t^2\psi + 2|\psi|^2\psi = 0. \qquad (1)$$

The medium (fiber) is imperfect, i.e. various macroscopic characteristics of the fiber fluctuate in space. Fluctuations of the dispersion coefficient, d, are believed to be one of the major sources of disorder present in real fibers [7]. This disorder is frozen, i.e. d is a random function of z. We assume that d fluctuates on short spatial scales and that the fiber is homogeneous on larger scales. The averaged value of d is a constant, which can be rescaled to unity by changing the units of t. One obtains, $d = 1 + \xi(z)$, where $\langle\xi\rangle = 0$. According to the Central Limit Theorem [8], ξ at the scales larger than the correlation length can be treated as a homogeneous Gaussian random process with zero mean, described by the quantity, $D = \int dz\langle\xi(z)\xi(z')\rangle$, which is the noise intensity. The pair correlation function of ξ is

$$\langle\xi(z_1)\xi(z_2)\rangle = D\delta(z_1 - z_2). \qquad (2)$$

We assume that the disorder is weak, i.e. $D \ll 1$.

At $z = 0$ a sequence of well-separated solitons is launched. In an ideal medium ($\xi = 0$) each of the solitons is preserved dynamically gaining, according to the exact single-soliton solution of Eq. (1), only a multiplicative phase factor. Because the medium is imperfect, the solitons, perturbed by impurities, shed radiation. The first problem is to describe the radiation. The soliton looses energy shedding radiation. Another problem is to describe the degradation of a single soliton. The tails of different solitons interfere with each other, forming a collective background. This fluctuating background affects all solitons. It results in the emergence of a long-range effective inter-soliton interaction, which is the final (but not the least) problem to be addressed. The long-range interaction dominates the direct interaction due to overlapping of soliton tails, as this direct interaction decays exponentially with separation [9, 10]. The emergence of the long-range interaction between the imperfect solitons in the pure ($\xi = 0$) NLS, mediated by the emitted radiation, was noted in [11]. The description of the calculation details, only briefly explained here, will be published elsewhere.

To examine the effects one should separate the degrees of freedom explaining solitons themselves and their continuous spectrum (radiation). For a single soliton this can be done as follows

$$\psi = \eta \exp\left(i\int_0^z \eta^2 dz\right)\left\{\frac{\exp[i\alpha + i\beta(t-y)]}{\cosh[\eta(t-y)]} + v\right\}, \quad (3)$$

where the four variables $\eta(z), \alpha(z), y(z), \beta(z)$ are the amplitude, phase, position, and the phase velocity of the soliton, respectively, and $v(t;z)$ stands for the continuous spectrum. The function v can be expanded in a complete set of delocalized eigen functions of the unperturbed ($\xi = 0$) NLS Eq.(1) linearized on the background of the perfect soliton [12, 13]. The continuous spectrum is separated by a gap in frequency from the four zero modes, associated with variations of η, α, y, β. The zero modes are localized in t. If D is finite but small, the four parameters vary slowly with z in contrast to the fast fluctuations of v, which are also small in amplitude. The separation of the slow and fast variables is the heart of the adiabatic approximation [12 – 16] which we explore here.

For a single soliton, the parameters y and β, which are assumed to be zero initially, cannot change due to the $t \to -t$ symmetry of Eq.(1). Thus, only two out of four soliton variables, η and α, evolve. The phase α is influenced by the noise ξ directly, $\partial_z \alpha = -\xi$, whereas η is affected by the noise indirectly, through the radiation shed by the soliton. Substituting Eq.(3) into Eq.(1) and keeping terms only linear in v and ξ, one arrives at an inhomogeneous equation for v with a source term proportional to ξ. The source is localized on the soliton. Solving the equation and averaging the result over the statistics of ξ, one deduces the expression

$$\langle |v|^2 \rangle \approx \frac{\pi}{16} D \eta^4 \ln\left[\frac{z\eta}{t}\right], \quad (4)$$

valid for $z\eta \gg t \gg 1$. Eq.(4) describes the extended radiation tails shed by the soliton due to the medium imperfectness. One observes a very slow decay of the radiation intensity in t. Eq.(4) is applicable at whatever large z (the soliton is always well distinguishable from the radiation). To disclose z-dependence of η, one can use the conservation of the integral $\int dt\, |\psi|^2$, $2\eta + \int dt\, \eta^2 |v|^2 = 2$. It shows that variations of η emerge in the second order in v. At $z \gg 1$ the quantity $\int dt |v|^2$ is self-averaged. Therefore, $|v|^2$ in the integral relation can be replaced by its average value, which is a function of η, according to Eq.(4). The result is a closed equation for η, and, finally, the solution

$$\eta(z) = (1 + 8Dz/3)^{-1/5}. \quad (5)$$

One concludes that the shedding soliton amplitude, η, remains unchanged until z reaches the scale z_η, $z_\eta = 1/D$.

Let us proceed to the multi-soliton case. A qualitatively new effect, associated with interaction of the shedding solitons through their radiation, emerges here. (The effect can be compared with the Van-der-Waals interaction, although the later is mediated by virtual photons whereas the inter-soliton interaction is due to real radiation.) The soliton positions are the first among other soliton parameters to be affected by the interaction. An essential change in the positions takes place at scales much shorter than z_η, where the soliton amplitudes are unchanged (still $z \gg 1$). This enables us to seek a solution of Eq.(1) in the form

$$\psi = \exp(iz)\left\{\sum_m \frac{\exp[i\alpha_m + i\beta_m(t-y_m)]}{\cosh[t-y_m]} + v\right\}, \quad (6)$$

where each term in the sum corresponds to a soliton, and v describes the continuous spectrum. One can derive equations for the soliton parameters, α_m, β_m, y_m, making use of the adiabatic approximation. The continuous spectrum is to be studied by substituting Eq.(6) into Eq.(1), and its subsequent linearization with respect to v and ξ. Equations for the soliton parameters are derived from Eq.(1) in the second order in v. Furthermore, as in the single-soliton case, $\int dt |v|^2$ is self-averaged, and therefore can be replaced by its noise average, which is a function of the soliton parameters. The resulting equations describing the slow dynamics of β_m and y_m are

$$\partial_z y_m = 2\beta_m, \qquad \partial_z \beta_m = F_m, \quad (7)$$

$$F_m = -\frac{\pi D}{36}\sum_{j,n} \cos(\alpha_j - \alpha_n)(y_j + y_n - 2y_m)^{-1}, \quad (8)$$

where the $j = n = m$ contribution has to be excluded from the sum. It is assumed in Eq.(8) that all the triple combinations, $|y_j + y_n - 2y_m|$, are large. The phase velocities, β_m, which are zero initially, remain small, $\sim Dz \ll 1$, and their effects on the continuous spectrum can be neglected. In spite of this smallness, the β terms in the equation for y give the major, $O(Dz)$, contribution (dominating the one proportional to $|v|^2 \sim O(D)$, omitted in Eq.(7) for β). The direct contributions from the noise to the absolute phase, which is $O(\xi)$, cancel out from the phase differences in Eq.(8). Other changes in the phases are not essential for $z \ll 1/D$.

The two-soliton version of Eqs.(7), (8) reads

$$\partial_z^2 x = \frac{\pi D}{18x}(1 + 4\cos\alpha), \quad (9)$$

where $\alpha = \alpha_1 - \alpha_2$ is the phase mismatch between the solitons and $x = y_2 - y_1$ stands for their relative separation. Eq.(9) describes the long-range interaction between the solitons. The α-dependence in Eq.(9) originates from the interference of the radiated waves with the same wavelengths, moving in opposite directions (in other words, joint radiation of the system of two solitons is not just a sum of the two single-soliton contributions). Notice that similar interference leads to the Anderson localization in $1d$ random media [17]. The sign of the interaction is controlled by the phase mismatch α: solitons repel each other if $0 < |\alpha| < \alpha_c \approx 1.823$, while $\alpha_c < |\alpha| < \pi$ corresponds to the solitons' attraction. The picture is opposite here to the one explaining the direct interaction of solitons, where the attraction at $\alpha = 0$ changes to repulsion at $\alpha = \pi$ [9, 10]. The solution of Eq.(9) with the condition $\partial_z x(0) = 0$ is given by

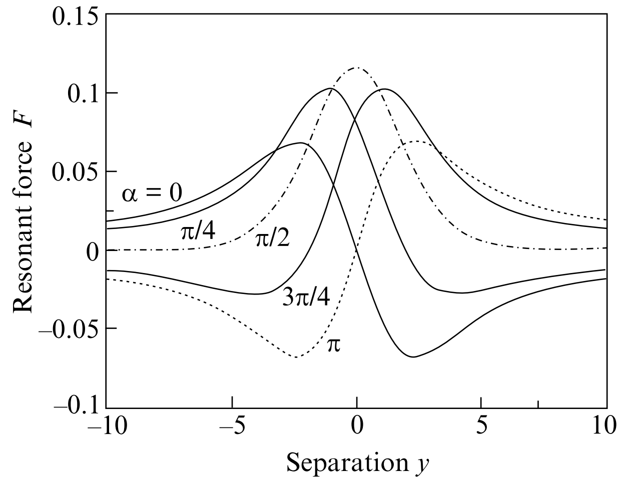
$$x(0)\mathrm{Erfi}\left\{\sqrt{\ln\left[x/x(0)\right]}\right\} = z\sqrt{(1+4\cos(\alpha))D/9}, \quad (10)$$

where Erfi is the imaginary error function. One finds that x changes on the order of its initial value at $z \sim z_{\text{int}} = x/\sqrt{D}$. Therefore, the scale separation, $z_{\text{int}} \ll z_\eta$, assumed in the derivation of Eqs.(7)–(10), is justified.

The inter-soliton interaction, described by Eqs.(7), (8), is triple. One may expect that a new physics, missed in the consideration of a soliton pair, would show itself in the more complex three-soliton case. A special, extremely long range, resonant interaction is indeed emerging here if the triple combination, $y \equiv y_1 + y_3 - 2y_2$, is $O(1)$, in spite of the fact that all the pair separations, $x_{jm} = y_m - y_j$, in the triad are large, $|x_{jm}| \gg 1$. The resonant contribution to the inter-soliton force (8), acting on the soliton positioned at y_2, is given by

$$F = \frac{\pi^2 D}{18}\,\mathrm{Im}\int\limits_0^\infty \frac{dq\, q(1+q^2)^2 e^{i\alpha - iqy}}{\cosh^2(\pi q/2)\sinh(\pi q)}, \quad (11)$$

where the ordering, $y_1 < y_2 < y_3$, is assumed. At $x_{13} \gg y \sim 1$, the resonant term, which is $O(1)$, dominates the non-resonant one, which is $O(1/x)$. The dependence of the resonant force on y for different values of the phase mismatch $\alpha = \alpha_1 - \alpha_3$ is shown in Figure. One observes that the middle soliton (positioned at y_2) is stable ($F'(y)$ is negative at the node position y_0, given by $F(y_0) = 0$) if $|\alpha| < \pi/2$, and unstable otherwise. The stability implies dynamical oscillations of the middle soliton around the stable node y_0, with a period in z estimated by $z_{\text{osc}} \sim 1/\sqrt{D}$. The period of the oscillations is still much shorter than, $z_{\text{int}} \sim x_{13}/\sqrt{D}$, where the size of the triad (x_{13}) changes on the order of its initial value. At $z \sim z_{\text{int}}$ the triad extends (or contracts, depending on the phases) as a whole under



Three-soliton resonant force $F(y)$ measured in the units of D

the action of the $\sim 1/x_{13}$ interaction, still keeping the relative positions of the solitons within the triad intact. The unstable case, which takes place if $\pi/2 < |\alpha| < \pi$, corresponds to the uncertainty of the relative positions of solitons within the triad at $z \sim z_{\text{int}}$. Figure also shows that the position of the node y_0 depends on the phase mismatch, α.

In the multi-soliton case the dynamics of pattern is controlled by Eq.(7), (8), provided all denominators are large. However, resonant configurations are possible here as well. Each of the configurations corresponds to a set of three solitons positioned according to $y_1 + y_3 - 2y_2 = O(1)$. The other solitons displayed inside the resonant pattern (in between y_1 and y_3) do affect the resonant interaction, i.e. it changes the force acting on the soliton positioned at y_2. If the difference between the number of solitons in between y_1, y_2, and y_2, y_3, respectively, is n, the expression Eq.(11) is modified via the multiplier, $[(q+i)/(q-i)]^{2n}$, inserted into the integrand.

Let us summarize the fundamental features of the interaction between the solitons through their radiation induced by disorder. First of all, the weakness of disorder, $D \ll 1$, allows us to reduce the original field problem, given by Eq.(1), to the N-body one, described by Eqs.(7), (8). Also, in spite of the stochastic nature of the original problem the N-body problem is *deterministic*. This is a consequence of self-averaging nature of the radiation intensity, $|v|^2$. Second, the interaction between the solitons through their radiation is not pair-wise. It is seen, in particular, through the triple character of the force F_m driving the β_m change in Eq.(7). (Each term in Eq.(8) corresponds to a contribution from a triad of solitons.) Third, the interaction in Eq.(8) for $x \gg 1$, is generically algebraic, i.e long-range. Fourth, not all the triple configurations contribute $O(D/x)$ into F_m; contribution from a resonant triad with $y \equiv y_j + y_k - 2y_m \sim 1$, is $O(D)$. Finally, the interaction is very sensitive to the soliton phases.

From the point of view of fiber optics applications the effect of the mutual interactions of the shedding solitons mediated by their radiation is really strong and potentially destructive. (The major requirement there is to preserve relative separations between solitons, which are bits of information and not to allow the solitons to leave their allocated time slots.) However, there exists another side of the analysis which may actually help to cure the problem. The dynamics is very sensitive to the values of the relative phases and positions in the soliton sequence, and there is certainly a great potential for reducing the inter-soliton forces by calibrating the positions (within the allocated slots) and phases of the solitons.

Another, radical (pattern independent) way to improve characteristics of propagation through noisy lines, called the pinning method, was suggested recently [18]. The idea is to pin the integral dispersion, $\int dz\, \xi$, to zero by inserting periodically short spans of fiber with carefully controlled dispersion. Let us now briefly explain how the pinning affects the phenomenon introduced in the letter. The pinning is effective if the pinning period, l, is shorter than all other scales, i.e. $l \ll 1$. Then, on the larger scales, the effective noise, $\xi_{\rm pin}$, is described by

$$\langle \xi_{\rm pin}(z_1)\xi_{\rm pin}(z_2)\rangle = -\frac{Dl^2}{3}\delta''(z_1 - z_2). \qquad (12)$$

Pinning of the noise leads to modification of the soliton degradation law Eq. (5), $\eta_{\rm pin} = (1 + 64Dl^2z/315)^{-1/9}$. The interaction of solitons is reduced by pinning. It is displayed through renormalization of D, $D \to Dl^2\eta^4/3$ in Eq. (4), and $D \to Dl^2/3$ in Eqs.(7), (10).

Let us emphasize that the phenomenon described in the letter is generic. Regardless whether it is additive or multiplicative frozen (t-independent) noise stimulates the shedding of radiation by solitons, which, in turn, mediates a long-ranged interaction between solitons. This long-ranged triple, and non-random character of the interaction, along with the sensitivity of the phenomenon to phases are generic features of any problem explaining the adiabatic evolution of solitons in the presence of induced radiation. However, if spatio-temporal (short-correlated both in t and z) noise is considered, the radiation effect, equivalent to the one considered in the letter, is masked by a jitter in relative soliton positions, $\delta y^2 \sim \tilde{D}z^3$ [19–21], where $\tilde{D}$ measures the intensity of the noise. Different solitons jitter independently, i.e. fluctuations of inter-soliton separations are described by the same δy. This spatio-temporal jitter is effective at the scales, $\sim \tilde{D}^{-1/3}$, where the long-range interaction of solitons mediated by radiation (phenomenon equivalent to the one considered in the letter)is still not essential.

The algebraic, $\sim 1/x$, character of the interaction is closely related to the reflective-less feature of the continuous radiation scattering on the soliton. However, the scattering becomes reflective in some nonintegrable generalizations of Eq. (1), that are of physical importance. The reflectivity leads to essential changes in the properties of the radiation and of the inter-soliton interaction. The reported stochastic phenomena (along with other of the kind, caused by random birefreingence of the fiber [22] and multichannel interaction[1)]) plays an important role in fiber communications.

We conclude the letter by brief discussion of real world parameters which would lead to the practical observation and system impact of the predicted effects in fiber optics communication. It was reported in [7] that fluctuations of the dispersion coefficient in a sample of the "dispersion shifted" fiber are of the order of its average value, i.e. $\sim 1\,\mathrm{ps/nm\cdot km}$, while the typical scale of the variations in dispersion is estimated from above by 1 km (the actual correlation scale is, probably, defined by linear dimensions of the devices used in the fiber production, i.e. it is somehow shorter, $\sim 100\,\mathrm{m}$). Therefore, for the pulse width of $\sim 7\,\mathrm{ps}$ (that corresponds to 28 Gb/s single-channel transmission rate) and the pulse period of, $\sim 50\,\mathrm{km}$, D is estimated by $10^{-3} - 10^{-2}$. Then, the soliton interaction is seen at $z_{\rm int} \sim (2{,}000 - 5{,}000)\,\mathrm{km}$, if solitons are separated by 5 soliton width. Notice, however, that decrease of the pulse width by factor q (correspondent to the factor q increase of the transmission rate) leads to the q^2 decrease of $z_{\rm int}$.

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