

On scaling fields in Z_N Ising models

V. A. Fateev⁺*, V. V. Postnikov[△], Y. P. Pugai⁺

⁺*L. D. Landau Institute for Theoretical Physics RAS, 117940 Moscow, Russia*

^{*}*Laboratoire de Physique Théorique et Astroparticules, Université Montpellier II, 34095 Montpellier, France*

[△]*Sochi's branch of People Friendship University, 354348 Sochi, Russia*

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We study the space of scaling fields in the Z_N symmetric models with the factorized scattering and propose simplest algebraic relations between form factors induced by the action of deformed parafermionic currents. The construction gives a new free field representation for form factors of perturbed Virasoro algebra primary fields, which are parafermionic algebra descendants. We find exact vacuum expectation values of physically important fields and study correlation functions of order and disorder fields in the form factor and conformal field theories perturbation approaches.

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1. Introduction. The calculation of exact correlation functions is one of the most complicated and interesting problems in the two dimensional integrable quantum field theory.

Remarkable solutions of this problem were found for the free field models including the Ising model without a magnetic field and for the conformal field theories (CFT) [1–3]. One of the difficulties in investigating massive integrable field theories with a non-trivial interaction is related with the fact that the space of scaling fields and, in particular, the space of different solutions of form factor equations, has no clear and simple description in terms of dynamical symmetry algebras like the free fermion algebra in the Ising model or Virasoro algebra in CFT. Existence of such infinite-dimensional algebras drastically simplifies the problem and allows applications of powerful algebraic methods. For instance in CFTs the Ward identities generated by the holomorphic stress energy tensor $T(z)$ permit to express the correlation functions of Virasoro descendant fields in terms of the correlators of the primary fields and to develop the free field approach for their calculation in the minimal models of CFT [2].

In massive integrable theories the situation with descendant fields is more difficult. Perturbing away from the conformal invariant point [4] one expects that the space of scaling fields is in a one-to-one correspondence with that of conformal fields. This statement was supported by several results on counting solutions of form factor equations [5, 6] in massive integrable field theories which demonstrate appearance of conformal algebra characters even off criticality [7]. This phenomena may indicate existence of some deformed conformal al-

gebra action in the space of form factors which would allow to extract useful information on descendants from knowledge on primaries (see also recent works in this direction [8]).

In this work we study the structure of the space of form factors of scaling fields in the parafermionic CFT with the central charge

$$c = 2(N - 1)/(N + 2), \quad N = 2, 3, \dots, \quad (1)$$

perturbed by the first thermal operator [9, 10]. In the free field approach [11, 12] to form factors we develop further the idea of the paper [13] on relations between different form factors of the theory. Namely, we introduce actions of simplest parafermionic modes in the space of form factors by exploiting the deformed parafermionic symmetry [14] of the underlying lattice model [15]. We check consistency of our construction by numerical studying of correlation functions of the theory by using a combination of form factor and conformal perturbation theory approaches [16]. To do this we also compute new exact vacuum expectation values (VEVs) of physically important fields.

2. Space of states of parafermionic CFT [3]. Conformal field theories with the parafermionic symmetry describe self-dual critical points of Z_N ($N = 2, 3, \dots$) generalizations [3, 15] of the Ising model, the last one corresponds to the $N = 2$ case. In a continuum limit order parameters $\sigma_k(x)$ ($k = 1, \dots, N - 1$) determining long range correlations of spins have anomalous dimensions

$$2d_k = \frac{k(N - k)}{N(N + 2)}. \quad (2)$$

Their Z_N charge is equal to k , i.e., under the action of the Z_N symmetry they transform as

$$\sigma_k \rightarrow \omega^{kn} \sigma_k, \quad n \in Z. \quad (3)$$

Z_N models allow Kramers-Wannier symmetry and the theory also contains the disorder parameters μ_k ($k = 1, \dots, N-1$) with the same dimensions $2d_k$ at the self-dual point. Operators μ_k transform under the action of the dual \tilde{Z}_N symmetry as

$$\mu_k \rightarrow \omega^{kn} \mu_k, \quad n \in Z. \quad (4)$$

All other fields naturally separating into families with the fixed $Z_N \times \tilde{Z}_N$ charges (k, l) behave as

$$\phi \rightarrow \omega^{kn+ln'} \phi, \quad n, n' \in Z.$$

The parafermions ψ_k and $\bar{\psi}_k$ generalizing usual fermions appear in the operator product expansions (OPE) of the order and disorder fields

$$\begin{aligned} \sigma_k(z, \bar{z}) \mu_k(0, 0) &= |z|^{-4d_k} z^{\Delta_k} [\psi_k(0) + \dots], \\ \sigma_k(z, \bar{z}) \mu_k^+(0, 0) &= |z|^{-4d_k} \bar{z}^{\Delta_k} [\bar{\psi}_k(0) + \dots]. \end{aligned} \quad (5)$$

These currents are holomorphic and generate the infinitely dimensional symmetry due to the conservation laws

$$\partial_{\bar{z}} \psi_k = 0, \quad \partial_z \bar{\psi}_k = 0. \quad (6)$$

We concentrate on the simplest solution of the associativity condition for the operator algebra of currents

$$\psi_k(z) \psi_l(0) = z^{\Delta_{k+l} - \Delta_k - \Delta_l} [\psi_{k+l} + \dots], \quad (7)$$

which corresponds to the conformal field theory with the central charge (1) and the conformal dimensions

$$\Delta_k = \frac{k(N-k)}{N}. \quad (8)$$

The fields $\psi_1 \equiv \psi$ ($\bar{\psi}_1 \equiv \bar{\psi}$) are the basic ones in the parafermionic algebra. It will be convenient for us to consider as well conjugate currents $\psi^\dagger \equiv \psi_{N-1}$ ($\bar{\psi}^\dagger \equiv \bar{\psi}_{N-1}$).

In the conformal models the space of states splits naturally into a direct sum of subspaces with the specified $Z_N \times \tilde{Z}_N$ charge (k, l)

$$\{F\} = \oplus \{F\}_{[m, \bar{m}]}, \quad N \geq m, \bar{m} \geq 1 - N, \quad (9)$$

where $[m, \bar{m}] = [k+l, k-l]$, $m + \bar{m} \in 2Z$. In these notations parafermionic currents and order-disorder fields belong to the following subspaces

$$\begin{aligned} \psi &\in \{F\}_{[2,0]}, & \psi^\dagger &\in \{F\}_{[-2,0]}, \\ \bar{\psi} &\in \{F\}_{[0,2]}, & \bar{\psi}^\dagger &\in \{F\}_{[0,-2]}, \\ \sigma_k &\in \{F\}_{[k,k]}, & \mu_k &\in \{F\}_{[k,-k]}. \end{aligned} \quad (10)$$

Conformal fields are conveniently classified according to the representations of the parafermionic algebra. The action of the parafermionic generators A_ν (A_ν^\dagger) is defined by the OPE

$$\begin{aligned} \psi(z) \phi_{[m, \bar{m}]} &= \sum z^{-\frac{m}{N} + n - 1} A_{\frac{1+m}{N} - n} \phi_{[m, \bar{m}]}, \\ \psi^\dagger(z) \phi_{[m, \bar{m}]} &= \sum z^{\frac{m}{N} + n - 1} A_{\frac{1-m}{N} - n}^\dagger \phi_{[m, \bar{m}]}. \end{aligned} \quad (11)$$

Notice that, if $\phi_{[m, \bar{m}]} \in F_{[m, \bar{m}]}$ has the conformal dimensions (d, \bar{d}) then the conformal dimensions of fields

$$\begin{aligned} A_\nu \phi_{[m, \bar{m}]} &\in \{F\}_{[m+2, \bar{m}]}, \\ A_\nu^\dagger \phi_{[m, \bar{m}]} &\in \{F\}_{[m-2, \bar{m}]}, \end{aligned} \quad (12)$$

are $(d - \nu, \bar{d})$.

Basic fields μ_k (σ_k) are the primaries of the parafermionic algebra. For instance, the following equations hold for $n \geq 0$

$$\begin{aligned} A_{\frac{1+k}{N} + n} \mu_k &= A_{\frac{1-k}{N} + n + 1}^\dagger \mu_k = 0, \\ \bar{A}_{\frac{1-k}{N} + n + 1} \mu_k &= \bar{A}_{\frac{1+k}{N} + n}^\dagger \mu_k = 0. \end{aligned} \quad (13)$$

All other fields of the model are obtained by the action of the currents $\psi, \bar{\psi}$ on μ_k (σ_k). Thus, the space of states of the conformal field theory decomposes into a direct sum of irreducible representations of the parafermionic algebra

$$\{F\} = \oplus_{k=0}^{N-1} [\mu_k]_A, \quad (14)$$

with the disorder operators μ_k playing the role of the highest weight fields.

However, from the viewpoint of Virasoro algebra, each of the spaces $[\mu_k]_A$ expands into a direct sum of representations with the highest weight fields $\phi_{m, \bar{m}}^k$. These Virasoro primaries are, in general, descendants in the parafermionic algebra representation. Indeed, let us denote as $(\psi^\dagger)^l \mu_k$ the field with the minimal conformal dimension that can be obtained by l -times application of the parafermionic generators ψ^\dagger to μ_k (and respectively for $\bar{\psi}$)

$$(\psi^\dagger)^l \mu_k = A_{\frac{2l-1-k}{N}}^\dagger A_{\frac{2l-3-k}{N}}^\dagger \cdots A_{\frac{1-k}{N}}^\dagger \mu_k. \quad (15)$$

Its conformal dimensions are easily computed from (2). Then, up to a normalization, the following relations take place

$$\begin{aligned}\phi_{k-2l, -k+2\bar{l}}^k &= (\psi^+)^l (\bar{\psi})^{\bar{l}} \mu_k, \quad l, \bar{l} = 0, 1, \dots, k, \\ \phi_{k+2l, -k-2\bar{l}}^k &= (\psi)^l (\bar{\psi}^\dagger)^{\bar{l}} \mu_k, \quad l, \bar{l} = 0, 1, \dots, N-k.\end{aligned}\quad (16)$$

For example, the physically important thermal operators $\varepsilon_k = \phi_{0,0}^{2k}$ are among these primaries. Recall that ε_k are local with respect to all fields and have the conformal dimensions

$$D_k = k(k+1)/(N+2). \quad (17)$$

3. Form factors for perturbed primary fields.

We consider the perturbation of the parafermionic conformal field theory by the first energy operator

$$\mathcal{A} = \mathcal{A}_{CFT} + \lambda \int d^2x \varepsilon_1(x). \quad (18)$$

The resulting massive theory is again integrable and Z_N symmetric. Depending on the sign of λ the system is in the ordered or disordered phase. We fix $\lambda > 0$ phase where the \tilde{Z}_N symmetry is destroyed and vacuum expectation values of disorder operators are non-zero.

Alternatively, this theory can be described also as a model with factorized scattering of Z_N charged particles [9]. The particles $a \in \{1, \dots, N-1\}$ in the Z_N ($N = 2, 3, 4, \dots$) symmetric models have masses

$$M_a = M \frac{\sin(\pi a/N)}{\sin(\pi/N)}. \quad (19)$$

Note, that the antiparticle a^\dagger is identified with the particle $N-a$. The scattering matrix of the lightest particles $a = 1$ has the form

$$S_{11}(\beta) = \frac{\sinh(\beta/2 + i\pi/N)}{\sinh(\beta/2 - i\pi/N)}. \quad (20)$$

The S matrices for higher particles are also diagonal and can be extracted from S_{11} according to a standard bootstrap prescription [9].

We follow the algebraic approach [11, 12, 14, 17] to the form factors originated from the analysis of lattice ABF model [15]. This lattice model falls into the same universality class as the Z_N Ising model. In the corner transfer matrix approach its dynamical symmetry is a deformation of a parafermionic symmetry. The deformed currents $\Psi(\alpha)$, as well as deformed primaries $\Phi_m^k(\alpha)$ (which are primary for deformed Virasoro algebra [12]) act in the corner transfer matrix space of states. Note, that there is an essential difference between the symmetries in off-critical situation and in the CFT. Thus, there exists only one chiral algebra, the parameter α is related with the spectral parameter but not with the distance, etc. However, from the mathematical viewpoint, the analysis of ABF models, including fusions

rules, Z_N symmetry, the structure of irreducible representations etc., is very similar to the conformal situation (3)–(10). Respectively, as in CFT, the problem of the calculation of physical quantities reduces to studying of operators acting in the direct sum of representations of the deformed parafermionic algebra.

In particular, the form factors are matrix elements of local operators in the basis of asymptotic states formed by operators diagonalizing the Hamiltonian. A peculiar feature of the Z_N models is that the Hamiltonian is diagonalized by the symmetry generators Ψ (Ψ^\dagger) themselves. At the scaling limit the currents become particle creation operators \mathcal{B} (\mathcal{B}^\dagger) which in the free fields approach [11] can be described as following:

$$\begin{aligned}\mathcal{B}(\beta) &= \frac{e^{-\frac{\beta}{N}P_2}}{\sqrt{2 \sin \frac{\pi}{N}}} \sum_{a=\pm 1} a e^{\frac{\pi i}{2N}(P_1+P_2+1)a} \mathcal{Z}_a(\beta), \\ \mathcal{B}^\dagger(\beta) &= -\frac{e^{\frac{\beta}{N}P_2}}{\sqrt{2 \sin \frac{\pi}{N}}} \sum_{a=\pm 1} a e^{\frac{\pi i}{2N}(-P_1+P_2-1)a} \mathcal{Z}_a^\dagger(\beta).\end{aligned}\quad (21)$$

These linear combinations contain the exponentials of free bosons $\mathcal{Z}_\pm(\beta)$ and $\mathcal{Z}_\pm^\dagger(\beta)$ and the “zero modes” operators $P_{1,2}$. The operators $\mathcal{Z}_a(\beta)$ and $\mathcal{Z}_a^\dagger(\beta)$ are assumed to satisfy the Wick theorem

$$\begin{aligned}\langle\langle \mathcal{Z}_{a_1}(\beta_1) \dots \mathcal{Z}_{a_n}(\beta_n) \mathcal{Z}_{b_1}^\dagger(\beta'_1) \dots \mathcal{Z}_{b_m}^\dagger(\beta'_m) \rangle\rangle &= \\ = \prod_{i<j} \langle\langle \mathcal{Z}_{a_i}(\beta_i) \mathcal{Z}_{a_j}(\beta_j) \rangle\rangle \prod_{i<j} \langle\langle \mathcal{Z}_{b_i}^\dagger(\beta'_i) \mathcal{Z}_{b_j}^\dagger(\beta'_j) \rangle\rangle \times \\ \times \prod_{i,j} \langle\langle \mathcal{Z}_{a_j}(\beta_j) \mathcal{Z}_{b_i}^\dagger(\beta'_i) \rangle\rangle,\end{aligned}\quad (22)$$

with the following contraction rules (where $\beta = \beta_1 - \beta_2$):

$$\begin{aligned}\langle\langle \mathcal{Z}_a(\beta_1) \mathcal{Z}_b(\beta_2) \rangle\rangle &= \langle\langle \mathcal{Z}_{-a}^\dagger(\beta_1) \mathcal{Z}_{-b}^\dagger(\beta_2) \rangle\rangle = \\ = \zeta(\beta) \frac{\sinh(\frac{\beta}{2} + \frac{i\pi}{2N}(a-b))}{\sinh \frac{\beta}{2}}, \\ \langle\langle \mathcal{Z}_a(\beta_1) \mathcal{Z}_b^\dagger(\beta_2) \rangle\rangle &= \langle\langle \mathcal{Z}_{-a}^\dagger(\beta_1) \mathcal{Z}_{-b}(\beta_2) \rangle\rangle = \\ = \zeta^\dagger(\beta) \cosh(\frac{\beta}{2} - \frac{i\pi}{2N}(a+b)).\end{aligned}\quad (23)$$

The functions $\zeta(\beta)$, $\zeta^\dagger(\beta)$ and their basic properties are given in the appendix A.

The perturbed primary fields $\phi_{m,-m}^k$ corresponds to the projectors to the states $\langle\langle \Phi_{-m}^k \rangle\rangle$ and $|\Phi_m^k\rangle$. We treat the last ones as states created by the deformed operators $\Phi_{\pm m}^k$ from the vacuum vector [17]

$$\begin{aligned}|\Phi_m^k\rangle &= \lim_{\alpha \rightarrow \infty} \Phi_m^k(\alpha)|0\rangle, \\ \langle\langle \Phi_{-m}^k \rangle\rangle &= \lim_{\alpha \rightarrow -\infty} \langle\langle 0 | \Phi_{-m}^k(\alpha) \rangle\rangle.\end{aligned}\quad (24)$$

Let us note that these bra and ket vectors in the corner transfer matrix picture correspond to the Virasoro algebra bra states created by left and right chiral parts of fields $\phi_{m,-m}^k$. The action of the zero modes on the vectors (24) reads as

$$P_1 |\Phi_m^k\rangle = k |\Phi_m^k\rangle, \quad P_2 |\Phi_m^k\rangle = -m |\Phi_m^k\rangle, \quad (25)$$

and we assume that the operators P_j commute with operators $\mathcal{Z}_a(\beta)$, $\mathcal{Z}_a^\dagger(\beta)$. The normalization of vectors $|\Phi_m^k\rangle$ is chosen according to the prescription [17]

$$\langle\langle \Phi_{-m}^k | \Phi_m^k \rangle\rangle = \langle \phi_{m,-m}^k \rangle, \quad (26)$$

which would correspond to the conformal normalization of scaling fields

$$\langle \phi(x) \phi(0) \rangle = |x|^{-4\Delta_\phi}, \quad x \rightarrow 0. \quad (27)$$

Then the form factors of the Virasoro algebra primary fields $\phi_{m,-m}^k$, which are local with respect to the fields μ_k , can be compactly written as following

$$\begin{aligned} & \langle \phi_{m,-m}^k | \{\beta\}, \{\beta'\} \rangle_{(n,n)} = \\ & = \langle\langle \Phi_{-m}^k | \prod_{j=1}^n \mathcal{B}(\beta_j) \prod_{j=1}^n \mathcal{B}^\dagger(\beta'_j) | \Phi_m^k \rangle\rangle. \end{aligned} \quad (28)$$

We implied here a shorthand notation for the state with n' lightest particles 1 and n antiparticles 1^\dagger

$$|\{\beta\}, \{\beta'\}\rangle_{(n,n')} \equiv |\beta_1, \dots, \beta_n, \beta'_1, \dots, \beta'_{n'}\rangle_{1^\dagger \dots 1^\dagger, 1 \dots 1}$$

The symbols of ordered products of particle creation operators stand for

$$\prod_{j=1}^n \mathcal{B}(\beta_j) = \mathcal{B}(\beta_1) \cdots \mathcal{B}(\beta_n).$$

The equation (28) is a natural generalization of the thermal operators form factors [14] to the $m \neq 0$ case. With the definitions (21)–(22) it has the conventional form

$$\begin{aligned} \langle \phi_{m,-m}^k | \{\beta\}, \{\beta'\} \rangle_{(n,n)} &= \frac{(-1)^n}{(2 \sin \frac{\pi}{N})^n} e^{\frac{\pi}{N} \sum (\beta_j - \beta'_j)} \times \\ & \times \sum_{\{a_j, b_j\}} \prod_j a_j b_j e^{\frac{i\pi}{2N} ((k+1)(a_j - b_j) - m(a_j + b_j))} \times \\ & \times \langle\langle \mathcal{Z}_{a_1}(\beta_1) \cdots \mathcal{Z}_{b_n}^\dagger(\beta'_n) \rangle\rangle \langle\langle \Phi_{-m}^k | \Phi_m^k \rangle\rangle. \end{aligned} \quad (29)$$

The remarkable property of our theory is that form factors of perturbed Virasoro primaries $\phi_{m,\bar{m}}^k$ including (28) can be obtained form factors of fields μ_k by the equation analogous to (16). Indeed, for large α the operators $\Phi_{\pm m}^k(\alpha)$ from (24) behave in the conformal limit, roughly speaking, as chiral parts of primaries $\phi_{m,-m}^k(e^{-\alpha}, e^{-\bar{\alpha}})$. Respectively, the operators $\Psi(\alpha)$ in

CFT limit become the parafermionic currents $\psi(e^{-\alpha})$. Basing on this correspondence, we introduce the analogs of rules (11)–(16) for the action of the simplest Fourier modes of deformed currents Ψ (Ψ^\dagger) as

$$\begin{aligned} |\Phi_{m+2}^k\rangle &= |A_{\frac{1+m}{N}} \Phi_m^k\rangle = \lim_{\alpha \rightarrow \infty} e^{-\frac{1+m}{N}\alpha} \Psi(\alpha) |\Phi_m^k\rangle, \\ |\Phi_{m-2}^k\rangle &= |A_{\frac{1-m}{N}}^\dagger \Phi_m^k\rangle = \lim_{\alpha \rightarrow \infty} e^{-\frac{1-m}{N}\alpha} \Psi^\dagger(\alpha) |\Phi_m^k\rangle, \\ \langle\langle \Phi_{\bar{m}+2}^k | &= \langle\langle \bar{A}_{\frac{1+\bar{m}}{N}} \Phi_{\bar{m}}^k | = \lim_{\alpha \rightarrow -\infty} e^{\frac{1+\bar{m}}{N}\alpha} \langle\langle \Phi_{\bar{m}}^k | \Psi(\alpha), \\ \langle\langle \Phi_{\bar{m}-2}^k | &= \langle\langle \bar{A}_{\frac{1-\bar{m}}{N}}^\dagger \Phi_{\bar{m}}^k | = \lim_{\alpha \rightarrow -\infty} e^{\frac{1-\bar{m}}{N}\alpha} \langle\langle \Phi_{\bar{m}}^k | \Psi^\dagger(\alpha). \end{aligned}$$

Since the operators \mathcal{B} (\mathcal{B}^\dagger) are, essentially, the scaling limits of the deformed parafermions Ψ (Ψ^\dagger), we put

$$\langle A_{\frac{1+m}{N}} \phi_{m,\bar{m}}^k | \{\beta\}, \{\beta'\} \rangle_{(n,n')} = \quad (30)$$

$$= \lim_{\alpha \rightarrow \infty} e^{-\frac{1+m}{N}\alpha} \langle\langle \Phi_{\bar{m}}^k | \prod_{j=1}^n \mathcal{B}(\beta_j) \prod_{j=1}^{n'} \mathcal{B}^\dagger(\beta'_j) \mathcal{B}(\alpha) | \Phi_m^k \rangle\rangle,$$

$$\langle \bar{A}_{\frac{1+\bar{m}}{N}} \phi_{m,\bar{m}}^k | \{\beta\}, \{\beta'\} \rangle_{(n,n')} = \quad (31)$$

$$= \lim_{\alpha \rightarrow -\infty} e^{\frac{1+\bar{m}}{N}\alpha} \langle\langle \Phi_{\bar{m}}^k | \mathcal{B}(\alpha) \prod_{j=1}^n \mathcal{B}(\beta_j) \prod_{j=1}^{n'} \mathcal{B}^\dagger(\beta'_j) | \Phi_m^k \rangle\rangle.$$

Starting from (28) we have the Z_N neutrality condition $2(n' - n) = m + \bar{m} + 2$. For Fourier modes A^\dagger , we obtain

$$\langle A_{\frac{1-m}{N}}^\dagger \phi_{m,\bar{m}}^k | \{\beta\}, \{\beta'\} \rangle_{(n,n')} = \quad (32)$$

$$= \lim_{\alpha \rightarrow \infty} e^{-\frac{1-m}{N}\alpha} \langle\langle \Phi_{\bar{m}}^k | \prod_{j=1}^n \mathcal{B}(\beta_j) \prod_{j=1}^{n'} \mathcal{B}^\dagger(\beta'_j) \mathcal{B}^\dagger(\alpha) | \Phi_m^k \rangle\rangle,$$

$$\langle \bar{A}_{\frac{1-\bar{m}}{N}}^\dagger \phi_{m,\bar{m}}^k | \{\beta\}, \{\beta'\} \rangle_{(n,n')} = \quad (33)$$

$$= \lim_{\alpha \rightarrow -\infty} e^{\frac{1-\bar{m}}{N}\alpha} \langle\langle \Phi_{\bar{m}}^k | \mathcal{B}^\dagger(\alpha) \prod_{j=1}^n \mathcal{B}(\beta_j) \prod_{j=1}^{n'} \mathcal{B}^\dagger(\beta'_j) | \Phi_m^k \rangle\rangle,$$

with the selection rule $2(n' - n) = m + \bar{m} - 2$.

These actions provide maps between solutions of form factor equations for fields with the different Z_N charges. Note, that the equation (16) holds for corresponding multi-particles form factors, as it can be directly verified from (28) and (30)–(33). In this way we extend free field formulae for primaries $\phi_{m,\bar{m}}^k$.

4. Short distance expansions and VEVs. As an application for new form factor formulae (28), (30)–(33) and for checking the consistency of the proposed parafermionic actions we analyze the correlation functions

$$G_+(x) = \langle \sigma_1(x) \sigma_1^\dagger(0) \rangle, \quad G_-(x) = \langle \mu_1(x) \mu_1^\dagger(0) \rangle, \quad (34)$$

computing it in the form factor approach (long distance expansion) from one side and in the conformal perturbation theory method (short distance expansion)

$$G_\pm(x) = r^{-4d_1} \sum C_\pm^{A_l}(r) \langle A_l(0) \rangle, \quad r = |x|, \quad (35)$$

from the other side [16]. The structure functions $C_{\pm}^{A_i}(r)$ above allow expansions into perturbation series

$$C_{\pm}^{A_i}(r) = r^{2\Delta_{A_i}} \left(C_{\pm,(0)}^{A_i} + \lambda r^{2(1-D_1)} C_{\pm,(1)}^{A_i} + \dots \right),$$

where the coefficients $C_{\pm,(1)}^{A_i}$ can be expressed through the integrals of correlation functions in CFT.

In the first order approximation the main contributions to (35) come from the primaries $I, \varepsilon_1, \varepsilon_2$ and the W -algebra descendant field $E_1 = W_{-1}^{(3)} \overline{W}_{-1}^{(3)} \varepsilon_1$

$$C_{\pm}^I = 1 \pm \frac{\lambda \pi r^{2(1-2u)} (\gamma(u) \gamma(3u))^{1/2} \gamma(4u)}{2(1-4u)^2 \gamma^3(2u)}, \quad (36)$$

$$C_{\pm}^{\varepsilon_1} = \mp \frac{r^{4u} (\gamma(u) \gamma(3u))^{1/2}}{2\gamma(2u)} + \frac{\lambda \pi r^2 u^2 \gamma(4u)}{4\gamma^4(2u) \gamma^4(1-u)},$$

$$C_{\pm}^{\varepsilon_2} = \mp r^{12u} \frac{\lambda \pi r^{2(1-2u)} u^2 \gamma^3(u) (\gamma(5u))^{1/2}}{12(1+2u)^2 \gamma^2(2u) (\gamma(3u))^{1/2}},$$

$$C_{\pm}^{E_1} = -r^{2+4u} \frac{u(1-4u) (\gamma(u) \gamma(3u))^{1/2}}{(1+2u)(1-2u) \gamma(2u)}.$$

Here and later we denote

$$\gamma(z) = \frac{\Gamma(z)}{\Gamma(1-z)}, \quad u = \frac{1}{N+2}. \quad (37)$$

Let us recall that the mass M of the lightest particle and the coupling constant λ can be expressed via each other. The exact relation between them in the conformal normalization $\langle \varepsilon_1(x) \varepsilon_1(0) \rangle = |x|^{-4D_1}$ is known [18] to have a form

$$(2\pi\lambda)^2 = \kappa^4 (1-2u) \gamma(u) \gamma(3u), \quad (38)$$

$$\kappa = M \frac{\Gamma(2/N) \Gamma(1-1/N)}{\Gamma(1/N)}. \quad (39)$$

Finally, to compare the CFT perturbation result (35)–(36) with the form factor predictions we have to compute vacuum expectation values of operators [19] in the conformal normalization (27). For μ_k these important quantities can be derived either from the reflection relations [20, 21] or via the deformed vertex operators [17]

$$\langle \mu_k \rangle_s = \omega^{sk} \kappa^{2d_k} \times \exp \int \frac{dt}{t} \left(\frac{\sinh kut \sinh(N-k)ut}{\sinh t \tanh Nut} - 2d_k e^{-2t} \right). \quad (40)$$

Here the index $s = 0, \dots, N-1$ enumerates the vacuum states in the \tilde{Z}_N broken phase. (Note, that the correlation functions G_{\pm} do not depend on it.)

For the general case, $\phi_{m,-m}^k = (\psi^\dagger)^l (\overline{\psi})^l \mu_k$, where $m = k - 2l$, the vacuum expectation values of fields are elegantly expressed in terms of $\langle \mu_k \rangle_s$ as

$$\frac{\langle \phi_{m,-m}^k \rangle_s}{\langle \mu_k \rangle_s} = \left(\frac{\kappa}{Nu} \right)^{\frac{k^2-m^2}{2N}} \prod_{i=0}^{l-1} \tilde{\omega}^{2s} \frac{(i+1) \gamma(\frac{i+1}{N})}{(k-i) \gamma(\frac{k-i}{N})}. \quad (41)$$

In particular, VEVs of thermal fields $\varepsilon_k = (\psi)^k (\overline{\psi})^k \mu_{2k}$ correspond to the $m = 0$ case in the equation above:

$$\langle \varepsilon_k \rangle_s = \frac{\kappa^{2D_k} (k!)^2}{(Nu)^k (2k)!} \left(\frac{\gamma((2k+1)u)}{\gamma(u)} \right)^{1/2} \times \prod_{i=1}^k \frac{\gamma^2(\frac{i}{N})}{\gamma(\frac{2i}{N}) \gamma((2i+1)u)}. \quad (42)$$

Using the method developed in [22] we also obtained the exact result for the expectation values of the normalized descendent fields $E_k = W_{-1}^{(3)} \overline{W}_{-1}^{(3)} \varepsilon_k$

$$\frac{\langle E_k \rangle_s}{\langle \varepsilon_k \rangle_s} = \kappa^2 \frac{(N+2)^2 \Gamma^2(1 + \frac{k+1}{N}) \Gamma^2(1 - \frac{k}{N})}{2N \Gamma^2(1 - \frac{k+1}{N}) \Gamma^2(1 + \frac{k}{N})} \times \frac{\Gamma(1 + \frac{2k}{N}) \Gamma(1 - \frac{2k+2}{N})}{\Gamma(2 - \frac{2k}{N}) \Gamma(2 + \frac{2k+2}{N})}. \quad (43)$$

Collecting these results we can analyze the short and long distance behaviors of correlators. Figs.1,2 demonstrate typical results obtained by numerical computa-

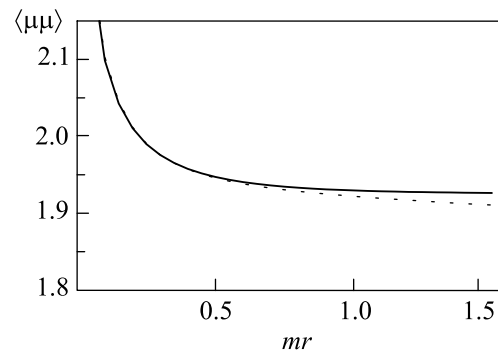


Fig.1. Disorder correlator $G_-(r)$. $N = 7$ Model

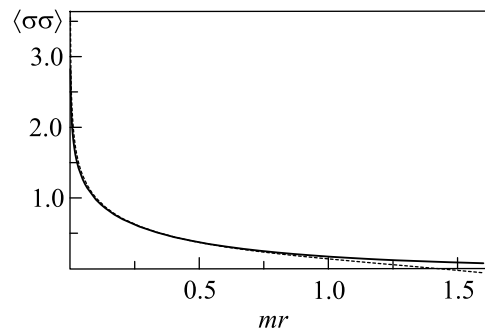


Fig.2. Order correlator $G_+(r)$. $N = 7$ Model

tions. The lower lines correspond to the short distance asymptotic from the CFT perturbation theory (35)–(36) and the upper lines give the form factor decomposition up to two particle. We see that both expansions are

in a perfect agreement at the intermediate distances and therefore provide good numerical data for G_{\pm} at all scales. This confirms the proposed identifications of form factors and also gives one of few examples of unitary theories where the correlators can be effectively studied in the scheme of Ref. [16].

5. Discussions. Algebraic relations among form factors of Z_N models can be extended for higher Fourier modes of parafermions as well. Though, in general, it is not clear how to confirm the proposals, we were able to provide an additional check supporting the validity of our construction. Namely, we verified the form factor counterpart of the quantum equations of motion, which look like:

$$\begin{aligned}\bar{\partial}\psi &= (1 - D_1)\lambda(\psi)\varepsilon_1, \\ \partial\bar{\psi} &= (1 - D_1)\lambda(\bar{\psi})\varepsilon_1.\end{aligned}\quad (44)$$

The form factors of operators $(\psi)\varepsilon_1$, $(\bar{\psi})\varepsilon_1$ are defined as in Eqs. (30)–(33), while to compute form factors of currents in the left hand side, we expanded corresponding matrix elements from the lattice theory [14] up to first non vanishing terms in the parameter of the lattice spacing. Equations (44) explain the relations between form factors of ψ and ε_1 originally found and discussed by Smirnov [13] for the Z_3 model.

Notice, that for small number of particles the form factors obtained from Eqs. (28), (30)–(33), (44) agree up to normalization factors with the results for Z_3 case [13] (see also Ref. [23]) and for Z_N case [24].

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A. Properties of two particle form factors. Let the contour C goes from infinity above the real axe, then around zero and then to infinity below the real axe. Introduce the notation

$$S_2(x) = \exp \frac{1}{2} \int_C \frac{dt}{2\pi it} \frac{\sinh(x - 2\pi)t}{\sinh^2(\pi t)} \log(-t). \quad (45)$$

Then the functions $\zeta^{(\dagger)}$ appearing in the equation (21) read

$$\begin{aligned}\zeta(\beta) &= \frac{i \sinh(\frac{\beta}{2})}{2 \sinh(\frac{\beta}{2} + \frac{i\pi}{N}) \sinh(\frac{\beta}{2} - \frac{i\pi}{N})} \times \\ &\times \frac{S_2(i\beta + 2\pi + \frac{2\pi}{N}) S_2(-i\beta + \frac{2\pi}{N})}{S_2^2(2\pi + \frac{2\pi}{N})},\end{aligned}\quad (46)$$

$$\zeta^{\dagger}(\beta) = \frac{1}{\cosh \frac{\beta}{2}} \frac{S_2^2(2\pi + \frac{2\pi}{N})}{S_2(i\beta + 3\pi + \frac{2\pi}{N}) S_2(-i\beta + \pi + \frac{2\pi}{N})}.$$

These functions satisfy the following important for us properties

$$\begin{aligned}\frac{\zeta(\beta)}{\zeta(-\beta)} &= S(\beta), & \frac{\zeta^{\dagger}(\beta)}{\zeta^{\dagger}(-\beta)} &= S^{\dagger}(\beta), \\ \zeta(\beta) &= \zeta(2\pi i - \beta), & \zeta^{\dagger}(\beta) &= \zeta^{\dagger}(2\pi i - \beta),\end{aligned}\quad (47)$$

$$\zeta(\beta)\zeta^{\dagger}(\beta + i\pi) = \left(i \sinh \left(\frac{\beta}{2} - \frac{i\pi}{N} \right) \right)^{-1},$$

$$\begin{aligned}&\prod_{j=1}^N \zeta \left(\beta + \frac{2j-1}{N} i\pi \right) = \\ &= \frac{S_2^{-2N}(2\pi + \frac{2\pi}{N})}{4 \sinh \left(\frac{\beta}{2} + \frac{i\pi}{2N} \right) \sinh \left(\frac{\beta}{2} - \frac{i\pi}{2N} \right)}.\end{aligned}$$

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