Pis'ma v ZhETF, vol. 83, iss. 7, pp. 326 – 330 © 2006 April 10

Liquid crystal defects and confinement in Yang-Mills theory

M. N. Chernodub[1)]

Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia

Submitted 21 February 2006

We show that in the Landau gauge of the $SU(2)$ Yang-Mills theory the residual global symmetry supports existence of the topological vortices which resemble disclination defects in the nematic liquid crystals and the Alice (half-quantum) vortices in the superfluid ^{3}He in the A-phase. The theory also possesses half-integer and integer-charged monopoles which are analogous to the point-like defects in the nematic crystal and in the liquid helium. We argue that the deconfinement phase transition in the Yang-Mills theory in the Landau gauge is associated with the proliferation of these vortices and/or monopoles.

PACS: 12.38.-t,12.38.Aw,61.30.-v

The confinement of color in Quantum Chromodynamics is one of the unsolved problems in the quantum field theory. Nowadays the commonly accepted point of view is that confinement is encoded in the dynamics of gluon fields which are described by the Yang-Mills (YM) theory. Popular approaches to the color confinement are based on the dual superconductor mechanism [1] and on the center vortex picture [2]. These approaches suggest that the confinement is caused by a particular dynamics of special gluon configurations called the Abelian monopoles and the central vortices, respectively. The formulation of these confinement mechanisms requires a gauge fixing of the non-Abelian symmetry which is also different for both approaches.

Recently, an interest to describe the confinement mechanism in the Landau gauge in terms of the particular (topological) configurations of the gluon fields has emerged [3, 4]. The Landau gauge is very attractive because it is very well studied both in perturbative and non-perturbative approaches. Moreover, this gauge is color-symmetric and it can be well formulated (up to the Gribov-copy problem which most of other gauges also have) both in the perturbative regime and in the continuum space-time. In this paper we propose explicit definitions and discuss the properties of the topological defects emerging in the Landau gauge.

We are working with the pure $SU(2)$ Yang-Mills theory in the Euclidean space-time. The Landau gauge is defined as a minimization of the functional,

$$\min_{\Omega} F[A^{\Omega}], \qquad F[A] = \int \mathrm{d}^4 x \, \left[A^a_\mu(x)\right]^2, \tag{1}$$

over the gauge transformations $\Omega \in SO(3)_{\mathrm{gauge}}$. The corresponding local differential gauge condition is

$$\partial_\mu A^a_\mu = 0. \tag{2}$$

The conditions (1), (2) fix the $SO(3)_{\mathrm{gauge}}$ *gauge* color freedom up to the $SO(3)_{\mathrm{global}}$ *global* color group,

$$SO(3)_{\mathrm{gauge}} \to SO(3)_{\mathrm{global}},$$

because the gauge-fixing functional (1) is invariant under the global (coordinate-independent) transformation,

$$A^a_\mu(x) \to \Omega^{ab}_{\mathrm{gl}} \, A^b_\mu(x), \qquad \Omega_{\mathrm{gl}} \in SO(3)_{\mathrm{global}}. \tag{3}$$

Since the gluon field does not transform under the action of the center $\mathbb{Z}_2$ of the gauge group $SU(2)$, the gauge symmetry of the *pure* $SU(2)$ Yang-Mills theory is in fact $SO(3) \sim SU(2)/\mathbb{Z}_2$.

The gauge field A_μ transforms in the fundamental representation of the residual $SO(3)_{\mathrm{global}}$ group (3). Let us construct the composite "color-spin" field

$$C^{ab}(x) = A^c_\mu(x) A^c_\mu(x) \cdot \delta^{ab} - A^a_\mu(x) A^b_\mu(x), \tag{4}$$

which is a scalar with respect to space-time rotations and a rank-2 tensor with respect to the global color transformations. The tensor field C transforms in the adjoint representation of the residual $SO(3)_{\mathrm{global}}$ group,

$$C(x) \to \Omega \, C(x) \, \Omega^T, \tag{5}$$

where the superscript T stands for the transposition in the color indices.

At this point it is convenient to introduce a mechanical interpretation of the matrix C^{ab}. The definition (4) is similar to the moment of inertia tensor of a solid body composed of mass-centers (labelled by the integer n) with the masses m^n located at the positions r_n:

$$I^{ij} = \sum_n m_n (r_n^2 \, \delta^{ij} - r^i_n r^j_n). \tag{6}$$

[1)]e-mail: Maxim.Chernodub@itep.ru

At each point x of the space-time the matrix C^{ab}, Eq. (4), corresponds to the inertia tensor of a "solid body" consisting of four ($\mu = 1,\dots,4$ in the four-dimensional space-time) mass-centers with equal "masses" $m_1 = \dots m_4 = 1$. The gauge field A^a_μ plays a role of a coordinate of the $\mu^{\rm th}$ mass-center. The color transformation (5) of the matrix C corresponds to a transformation of the inertia tensor (6) under a spatial rotation.

At each point x of the space-time the matrix $C^{ab}(x)$ can be diagonalized with the help of the $SO(3)$ transformation $\Theta(x)$,

$$C^{ab}(x) = \Theta(x)\,\mathrm{diag}(c_1, c_2, c_3)\,\Theta^T(x). \qquad (7)$$

The eigenvalues c_k, $k = 1,2,3$ can be interpreted as "moments of inertia" defined with respect to the orthonormal "principal axes of inertia" $\mathbf{e}_k$, $k = 1,2,3$, respectively. The axes $\mathbf{e}_k$ are normalized eigenvectors of the "moment of inertia tensor" C. The transformation $\Theta(x)$ relates the default basis in the color space with the basis of the principal axes of inertia at the point x. The eigenvalues of the matrix C are ordered, $c_1 \geqslant c_2 \geqslant c_3$, and below we assume that these eigenvalues are not degenerate unless it is stated otherwise.

To pursue this mechanical analogy further, the tensor C can be associated an "ellipsoid of inertia" in the color space. The directions of the axes of this ellipsoid coincide with principal axes of inertia $\mathbf{e}_k$, $k = 1,2,3$, while the axes lengths are proportional to the inverse square roots of the corresponding moments of inertia. The direction of the longest axis of the ellipsoid of inertia – corresponding to the lowest principal moment of inertia – defines a unit vector $\mathrm{n}(x) \equiv \mathbf{e}_3(x)$ in the color space.

The vector $\mathrm{n}(x)$ is direction-less since the vectors $\pm\mathbf{n}$ correspond to the same ellipsoid of inertia. Indeed, the ellipsoid is invariant under the transformations of the D_2 group which is a finite subgroup of the $SO(3)$ group of rotations. The elements of D_2 are π-rotations about any two principal axis of the ellipsoid. The product of these elements generate the π-rotation about the third axis. The π-rotation of the ellipsoid about either $\mathbf{e}_1$ or $\mathbf{e}_2$ axis leads to the flip $\mathbf{n} \to -\mathbf{n}$. Thus, the eigenvector corresponding to the lowest eigenvalue of the composite symmetric field (4) defines an arrowless vector in the color space $\mathbf{n}$ – or, in other words, a projective plane $RP(2)$ element – for a given configuration of the gauge fields A^a_μ.

The arrowless feature of the vector $\mathrm{n}(x)$ allows to formulate a liquid crystal-like structure in the YM fields. The spin field $\mathbf{n}$ may also be associated with the largest principal axes of the axially symmetric molecules (*i.e.*, which is also called the "director field") in nematic liquid crystals. The ordinary nematic crystals [5, 6] are liquids composed of rod-like molecules which are randomly oriented in the isotropic (high-temperature) phase. At low temperatures the systems goes into the liquid crystal phase in which the "rods" tend to align parallel to the direction $\mathbf{n}_0$ in macroscopic volumes. The molecules in liquid crystals do not have a positional order while being orientationally ordered.

The rod-like molecule in the nematic is invariant under (i) the $\mathbb{Z}_2$ group consisting of the π-rotations about any axis perpendicular to the longest axis of the molecule and (ii) the $SO(2)$ group of rotations about the longest axis. Thus, the physical space of the axial molecule – corresponding to the ellipsoid of inertia defined by the color tensor (4) – is the coset

$$G/H = SO(3)/(\mathbb{Z}_2 \times SO(2)). \qquad (8)$$

Let us imagine for a moment that we have integrated out in the YM partition function (fixed to the Landau gauge) all the degrees of freedom but the spins $\mathbf{n}$. Then at each point of the space-time the physical group of the $\mathbf{n}$-spins would become exactly the coset group (8). Thus, after all degrees of freedom but $\mathbf{n}$ are integrated out, the D_2 invariance group of the "ellipsoid of inertia" must be replaced by the $\mathbb{Z}_2 \times SO(2)$ group which leaves the arrowless vector $\mathbf{n}$ intact.

One should note that the coset group (8) does *not* correspond to the symmetry breaking $G \to H$ in the YM theory. In the pure YM theory the direction of the vector field $\mathbf{n}$ changes from point to point leading to the vanishing vacuum expectation value, $\langle \mathbf{n} \rangle = 0$ since the color symmetry is obviously not broken in this case. Equation (8) defines the group, the non-unit elements of which make the physically distinguishable changes to the director field $\mathbf{n}$.

Despite the absence of the symmetry breaking, still one can define defects associated with topological invariants of the arrowless vector $\mathbf{n}$. These definitions do not depends on the dynamics of the model, while they do depend on the possibility to define a unit vector with particular symmetry properties (8). The coincidence of the physical group (8) with the one of the nematic crystals [5, 7] allows us to discuss topological (with respect to the residual $SO(3)_{\rm global}$ group) defects in the Landau gauge of the YM theory using the corresponding condensed matter analogues. It it worth mentioning an attempt [8] to construct an effective nematic theory from the YM theory in an explicitly gauge-invariant form using the field-strength tensor of the $SU(2)$ gauge field. Unfortunately, the dynamics of these defects turned out to be very sensitive to the ultraviolet scale.

Topological defects are characterized by homotopy groups of the physical space (8). The π_0 group is trivial, $\pi_0(G/H) = 1$, telling us that there are no topologically stable domain walls made of the director field $\mathbf{n}$.

The first homotopy group is nontrivial, $\pi_1(G/H) = \mathbb{Z}_2$, indicating that there are topologically stable vortices characterized be the only nontrivial element of the $\mathbb{Z}_2$ group. The $\mathbb{Z}_2$-vortices are very well known as "disclination defects" in the physics of liquid crystals [5]. The typical example of the straight static vortex going perpendicularly to the $x - y$ plane is described by the director field of the form:

$$\mathbf{n}_{\text{vort}} = (\cos n\varphi, \sin n\varphi, 0) \qquad (9)$$

where φ is the azimuthal angle in the $x-y$ plane and n is the topological number which may take integer and half-integer values. The integer-values vortices are unstable, and the topologically stable elementary vortex has the fractional circulation number $n = 1/2$. In the center of the vortex the color tensor (4) must be at last double degenerate to support the two-dimensional hedgehog-like singularity (9).

The presence of the half-winding vortices is possible because the directions $\mathbf{n}$ and $-\mathbf{n}$ are physically indistinguishable. The director field $\mathbf{n}$ rotates in 180° degrees in the color space, (*i.e.*, $\mathbf{n} \to -\mathbf{n}$), as one makes the full 360°-turn around the vortex position. These vortices share similarities with the central vortices observed numerically in the Maximal Center gauge [2] of the YM theory, as well as with the so-called Alice strings discussed both in the particle physics [9] and in the condensed matter physics (*i.e.*, as half-quantum vortices which exist both in nematic liquid crystals [5, 6] and in A-phase of the superfluid ^{3}He [10, 11]).

The non-triviality of the second homotopy group, $\pi_2(G/H) = \mathbb{Z}$ guarantees the presence of the topologically stable monopoles. The monopoles are particle-like objects characterized by positive half-integer winding numbers $m = M/2$, $M \in \mathbb{Z}_+ \equiv \mathbb{Z}/\mathbb{Z}_2 = 0, 1, 2, \dots$, which count the number of sweeping of the spatial sphere into the sphere of the color group. Due to the $\mathbb{Z}_2$–identification $\mathbf{n} \to -\mathbf{n}$, the number m can be half-integer.

The monopole current can be formulated in terms of the winding number,

$$k_\mu = \frac{1}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu\alpha\beta}\frac{\partial n^a}{\partial x_\nu}\frac{\partial n^b}{\partial x_\alpha}\frac{\partial n^c}{\partial x_\beta}. \qquad (10)$$

The current is represented as δ-singularities on a set of closed loops $\mathcal{C}$ parameterized by the four-vector $X(\tau)$,

$$k_\mu(x) = \sum_{\mathcal{C}} m_{\mathcal{C}} \int_{\mathcal{C}} \mathrm{d}\tau \, \frac{\partial X^{\mathcal{C}}_\mu(\tau)}{\partial \tau}\, \delta^{(4)}(x - X^{\mathcal{C}}(\tau)), \quad (11)$$

where $m_{\mathcal{C}}$ is the charge ascribed to the loop $\mathcal{C}$. By construction (10), the monopole current is closed, $\partial_\mu k_\mu = 0$. A three-dimensional analogue of Eq. (10) was used in Ref. [12] to formulate π_2-singularities in the superfluid Helium and in the liquid crystals.

Due to the identification $\mathbf{n} \leftrightarrow -\mathbf{n}$ the monopole winding numbers $\pm m_{\mathcal{C}}$ correspond to the same monopole according to Eq. (10). This feature is perfectly consistent with the presence of the Alice $\mathbb{Z}_2$-vortices discussed above, because the monopole with the charge m transforms into the (physically the same!) monopole with the charge $-m$ after circling around the Alice string [11].

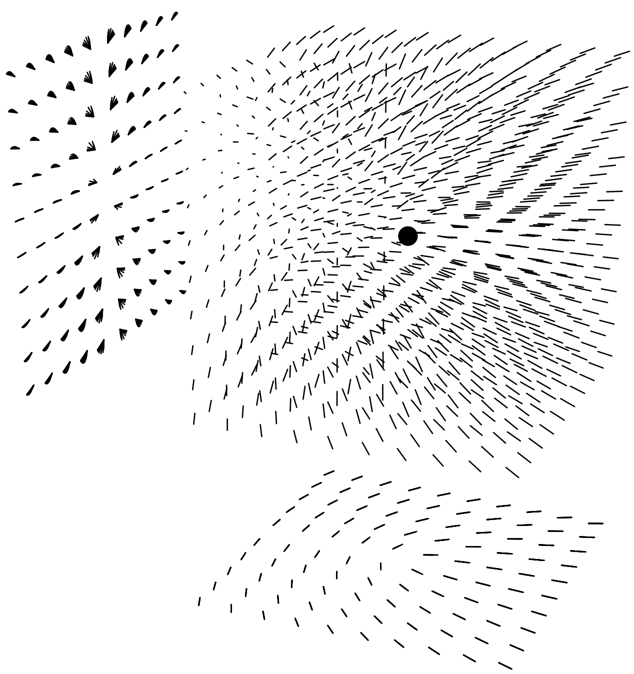
A class of the static unit director fields corresponding to the m-charged monopoles is given in Refs. [13, 12]:

$$\mathbf{n}_{\text{mon}} = (\sin\vartheta_m(\theta)\cos m\varphi, \sin\vartheta_m(\theta)\sin m\varphi, \cos m\varphi), \qquad (12)$$

where θ and φ are polar and azimuthal angles, and

$$\vartheta_m(\theta) = 2\arctan\{[\tan(\theta/2)]^m\}.$$

The positive topological number m may take integer and half-integer values. The monopole current (10) is $k_\mu(\mathbf{x}, t) = m\,\delta_{\mu,4}\,\delta^{(3)}(\mathbf{x})$. The static $m = 1/2$ monopole (12) is visualized in a spatial time-slice in Figure. The $m = 1$ monopole has a standard three-dimensional hedgehoglike structure, $\mathbf{n} = \hat{\mathbf{x}}$.



The schematic behavior of director field (12) for the $m = 1/2$ monopole. The projections of the director field $\mathbf{n}$ into two planes are also shown

The third homotopy group of the physical space is also non-trivial, $\pi_3(G/H) = \mathbb{Z}$, which leads to the appearance of the instantons in the four dimensional sense.

As an example of a particular configuration of the gluon fields, consider the BPS monopole solution [14] to the SU(2) Yang–Mills equation of motion:

$$A_i^a = \frac{1}{g} f(r)\varepsilon_{iab}\hat{x}^b, \qquad f(r) = \frac{1}{r}\Big(1 - \frac{r}{\sinh r}\Big), \quad (13)$$

$$A_4^a = \frac{1}{g} h(r)\,\hat{x}^a, \qquad h(r) = \frac{1}{r}\Big(r\coth r - 1\Big), \quad (14)$$

where $\hat{x}^a = x^a/|\mathbf{x}|$, and $r \equiv |x|$ is assumed to be scaled by an arbitrary factor, $r \to r_0 \cdot r$, to make a dimensionless quantity. This static configuration satisfies the local Landau gauge condition (2).

The color tensor C^{an}, Eq. (4), evaluated at the monopole configuration (13), (14) has three eigenvalues,

$$c_1 = c_2 = \frac{1}{g^2}\Big(f^2(r) + h^2(r)\Big), \quad c_3 = \frac{2}{g^2} f^2(r).$$

Since the two out of the three eigenvalues are degenerate, the "ellipsoid of inertia" is axially symmetric. Moreover, $f(r) < h(r)$ for $r > 0$, therefore $c_1(r) = c_2(r) > c_3(r)$, and the ellipsoid is always rod-like (contrasted to a disk-like form). The longest axis of the ellipsoid forms a hedgehog-like structure around the center of the monopole, $\mathbf{n} \equiv \mathbf{e}_3 = \hat{\mathbf{x}}$. Thus, in the Landau gauge the BPS YM-monopole can be identified with the $m = 1$ charge global monopole, which can be called as "arrowless hedgehog" due to the $\mathbb{Z}_2$ symmetry $\mathbf{n} \leftrightarrow -\mathbf{n}$.

Since the color symmetry is not broken in the YM theory the director field does not have a constant direction in the color space, $\langle \mathbf{n} \rangle = 0$. Therefore both the confinement and the deconfinement phases correspond – in the language of the nematic liquid crystals – to the (color) isotropic phases, where the density of the topological defects is non-zero due to large-angle fluctuations of the director field.

In the confinement phase a network of the defects is likely to be percolating so that any two infinitely-separated points have a finite probability to be connected by a world-trajectory of the defects. Generally, the percolation of a defect trajectory means an existence of a non-vanishing low-momentum (infrared) component of a corresponding field, which implies the condensation of the defect. The percolation (condensation) feature is shared by many realistic theories possessing the topological defects. The percolation happens in the phases where the symmetry group (responsible for the appearance these defects) is unbroken. The relevant examples are the symmetric phases of the Abelian Higgs (or, the Ginzburg–Landau) model [15] and the non-Abelian Higgs (or, the Standard Electroweak) model [16] where the monopole-like and vortex-like defects are known to be proliferating at infinitely large distances.

To discuss the behavior of the monopoles in the deconfinement phase we note, that both integer and half-integer monopoles in the Landau gauge are characterized by the specific behavior of the director field in a local three-dimensional time-slice perpendicular to the direction of the monopole current (11). In order to support a non-zero sphere-to-sphere winding number (10) the director field must evolve in all three dimensions in the local vicinity of the monopole. The hedgehog structures involving the Euclidean time direction are suppressed at high temperatures by high Matsubara frequencies, leading to a suppression of the monopoles moving in spatial directions.

Thus, at high temperatures the monopoles must be static, and, as a consequence, non-percolating. The high and the low temperature regimes must be separated by a percolation transition, which is very likely to at the same temperature as the deconfinement phase transition as it happens in the Abelian and non-Abelian Higgs theories [15, 16]. A similar property is also observed numerically for the Abelian monopoles in the Maximal Abelian gauge of the Yang-Mills theory [17].

Similar arguments can be applied to the half-quantum vortices in the Landau gauge: they are expected to be proliferating in the confinement phase and tend to be static in the deconfinement phase. Since the vortices are line-like defects, two different possibilities can be in the deconfinement phase: (i) the vortices may exist in the form of dominantly static and non-percolating short loops, (ii) or the static vortices may exist in the form of long loops leading to a spatial percolation of the vortex trajectories. The experience gained with the center vortices in the Maximal Center gauge [18] tells us that it is the first option that is likely to be realized in the high-temperature phase.

The dynamics of the topological defects is important because it is related to the confinement of color. In the confinement phase the percolating defects should cause a disorder which should lead to the area law of sufficiently large Wilson loops. In the deconfinement phase the defects are expected to be dominantly static and the disorder spreads only in spatial dimensions leading to the deconfinement of the static quarks and to confinement of the "spatial" quarks in a sense of the area-law for the spatial Wilson loops.

Let us discuss the (dis)order parameter phase transition within the described (nematic) picture. In the language of the topological defects, the disorder parameter for the deconfinement phase transition is given by the percolation probability for the monopoles (and, probably, vortices) to proliferate for infinitely long distances. This probability should be zero in the deconfinement phase and non-zero in the confinement phase.

In the language of fields a best candidate to discuss the transition might be the color tensor C^{ab}, Eq. (4), because the field C^{ab} (up to an inessential color-singlet part) looks similar to the diamagnetic susceptibility in the nematic liquid crystals. The diamagnetic susceptibility is known to be the order parameter of the nematic-isotropic phase transition in the liquid crystals [5]. In the nematic (low temperature) phase the director field is dominantly constant leading to an anisotropy in the diamagnetic susceptibility, while in the high temperature phase the director field is random and the diamagnetic susceptibility is isotropic. Since the color symmetry is unbroken in the YM theory, one must have $\langle C^{ab} \rangle \propto \delta^{ab}$ both in the confinement and in the deconfinement phases. Consequently, the field C^{ab} can not be used as an order parameter for the deconfinement phase transition.

Thus, the deconfinement phase transition in the YM must be associated with a transition from one isotropic phase of the nematic crystal to another isotropic phase. The two distinct isotropic phases were indeed observed in the lattice numerical simulations of the three-dimensional nematic crystals [20]. One of such isotropic phases is characterized the defect condensation. In the language of the YM theory it corresponds to the confinement phase. Another isotropic phase – called the topologically ordered phase [20] – is characterized by the absence of the condensate (the analog of the deconfinement phase in the YM theory). Surprisingly, the isotropic-to-isotropic transition in the nematic crystal is the second-order phase transition lying in the 3D Ising universality class [20]. The fact that the order and the universality class of the phase transition in the nematic crystal coincide with the order and the class of the finite-temperature phase transition in the SU(2) YM theory [21] provides an additional support in favor of the liquid crystal interpretation of the YM theory in the Landau gauge.

The author is supported by grants RFBR # 04-02-16079, RFBR # 05-02-16306, DFG grant 436 RUS # 113/739/0 and MK # 4019.2004.2. The author is grateful to F.V. Gubarev for useful discussions, and to A. Niemi for turning the attention of the author to Ref. [11]. The author is also thankful for the kind hospitality and stimulating environment to the members of the Theoretical Particle Physics group of Humboldt University (Berlin).

1. Y. Nambu, Phys. Rev. D **10**, 4262 (1974); G. 't Hooft, in *High Energy Physics*, Ed. A. Zichichi, EPS International Conference, Palermo (1975); S. Mandelstam, Phys. Rept. **23**, 245 (1976).
2. L. Del Debbio, M. Faber, J. Greensite, and S. Olejnik, Phys. Rev. D **55**, 2298 (1997); L. Del Debbio, M. Faber, J. Giedt et al., *ibid.* **58**, 094501 (1998); J. Greensite, Prog. Part. Nucl. Phys. **51**, 1 (2003).
3. T. Suzuki, K. Ishiguro, Y. Mori, and T. Sekido, Phys. Rev. Lett. **94**, 132001 (2005).
4. M. N. Chernodub, hep-th/0506107.
5. M. J. Stephen and J. P. Straley, Rev. Mod. Phys. **46**, 617 (1974); D. C. Wright and N. D. Mermin, Rev. Mod. Phys. **61**, 385 (1989).
6. P. G. de Gennes, *The Physics of Liquid Crystals*, Clarendon Press, Oxford, 1975.
7. L. Michel, Rev. Mod. Phys. **52**, 617 (1980).
8. F. V. Gubarev, unpublished (2003).
9. A. S. Schwarz, Nucl. Phys. B **208**, 141 (1982).
10. G. E. Volovik and V. P. Mineev, JETP **45**, 1186 (1977).
11. G.E. Volovik, *The Universe in a Helium Droplet*, Clarendon Press, Oxford, 2003.
12. S. Blaha, Phys. Rev. Lett. **36**, 874 (1976).
13. A. Saupe, Mol. Cryst. **21**, 211 (1973).
14. E.B. Bogomolny, Sov. J. Nucl. Phys. **24**, 449 (1976); M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975); P. Rossi, Phys. Rept. **86**, 317 (1982).
15. M. Chavel, Phys. Lett. B **378**, 227 (1996); M. Baig and J. Clua, Phys. Rev. D **57**, 3902 (1998); S. Wenzel, E. Bittner, W. Janke et al., Phys. Rev. Lett. **95**, 051601 (2005).
16. M.N. Chernodub, F.V. Gubarev, E.M. Ilgenfritz, and A. Schiller, Phys. Lett. B **443**, 244 (1998); *ibid* B **434**, 83 (1998).
17. T.L. Ivanenko, A.V. Pochinsky, and M.I. Polikarpov, Phys. Lett. B **302**, 458 (1993).
18. M.N. Chernodub, M.I. Polikarpov, A.I. Veselov, and M.A. Zubkov, Nucl. Phys. Proc. Suppl. **73**, 575 (1999).
19. M. Engelhardt, K. Langfeld, H. Reinhardt, and O. Tennert, Phys. Rev. D **61**, 054504 (2000).
20. P.E. Lammert, D.S. Rokhsar, and J. Toner, Phys. Rev. Lett. **70**, 1650 (1993).
21. B. Svetitsky, Phys. Rept. **132**, 1 (1986).