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Two-stream-like mechanism of zonal-flow generation by Rossby waves in shallow rotating fluid

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It is shown that the small-scale Rossby waves in shallow rotating fluid, placed in gravitational field, can generate large-scale zonal flows by means of a two-stream-like mechanism. This mechanism is revealed in the conditions when the Lighthill instability criterion is not satisfied.

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According to [1] and references therein, wave properties of shallow incompressible rotating fluid placed in gravitational field are similar to those of magnetized plasma. Such a similarity serves as a prerequisite for an interchange by conceptual and methodical achievements between the hydrodynamics and plasma physics. Historically, this interchange has started with establishment of the fact that the hydrodynamic solitary vortices observed in nature and in the laboratory can be considered as models of wave processes in magnetized plasma and that, vice versa, there exists a possibility of the plasma simulation of hydrodynamic vortices (see in detail [1 – 4]). Recently the interplay of hydrodynamics and plasma physics is shifted to the problem of large-scale zonal-flow generation by small-scale waves which, in the case of hydrodynamics, are the Rossby waves while, in the case of plasma physics, – the electrostatic drift waves [5 – 7].

Studying the zonal-flow generation by the Alfvén waves in magnetized plasma, [8] has pointed out that, in addition to the standard mechanism of such a generation, revealed in the single-pump-wave approximation (see, e.g., [9]), there is a new generation mechanism revealed in the case of two-humped spectra of the pump waves [9], which has been called in [8] the two-stream-like-mechanism. The goal of the present paper is to show that this zonal-flow generation mechanism is predicted also for the Rossby waves.

Let us start with the Charney-Obukhov equation for the Rossby waves in a shallow rotating fluid in a gravitational field. Turning to [1], this equation can be presented in the form (see also [5] presenting it in the dimensionless form)

$$\frac{\partial h}{\partial t} + V_R \frac{\partial h}{\partial y} - r_R^2 \left(\frac{\partial}{\partial t} + \mathbf{V}_D \cdot \nabla \right) \nabla_\perp^2 h = 0. \qquad (1)$$

Here $h = H/H_0$, H is the fluid depth, H_0 is the equilibrium fluid depth, $r_R = (gH_0)^{1/2} \big/ f$ is the Rossby-Obukhov radius, g is the gravitational force, f is the Coriolis parameter, $V_R = -(gH_0/f)\, d\ln f/dx$ is the equilibrium Rossby velocity, $\mathbf{V}_D = (g/f)[\nabla H \times \mathbf{z}]$ is the "cross-field drift velocity" components, y is the "drift" direction, x is the "radial" direction, i.e., the direction of fluid inhomogeneity, $\mathbf{z}$ is the unit vector along z.

We analyze our starting equation (1) by the convective-cell method [10]. We represent

$$h = \widetilde{h} + \widehat{h} + \overline{h}. \qquad (2)$$

Here $\widetilde{h}$, $\widehat{h}$, and $\overline{h}$ describe the primary modes, the secondary small-scale modes, and the zonal flow, respectively. The function $\overline{h}$ is taken in the form

$$\overline{h} = \overline{h}_0 \exp(-i\Omega t + iq_x x) + \text{c.c.}, \qquad (3)$$

where Ω and q_x are the frequency and radial mode number of zonal-flow, respectively, c.c. is the complex conjugative. The function $\widetilde{h}$ is presented as

$$\widetilde{h} = \sum_{\mathbf{k}} \left[\widetilde{h}_+(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) + \right. \\ \left. + \widetilde{h}_-(\mathbf{k}) \exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t) \right], \qquad (4)$$

where ω and $\mathbf{k}$ are the frequencies and wave vectors of the primary modes, $\widetilde{h}_-(\mathbf{k}) = \widetilde{h}_+^*(\mathbf{k})$, "$*$" means the complex conjugative, the summation is performed over all totality of the primary modes. At last, the function $\widehat{h}$ is given by

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$$\widehat{h} = \sum_{\mathbf{k}} \left[\widetilde{h}_+(\mathbf{k})\exp\left(i\mathbf{k}_+\cdot\mathbf{r} - i\omega_+ t\right) + \right.$$
$$\left. + \widetilde{h}_-(\mathbf{k})\exp\left(i\mathbf{k}_-\cdot\mathbf{r} - i\omega_- t\right) + \text{c.c.}\right]. \quad (5)$$

Here $\widehat{h}_\pm(\mathbf{k})$ are the side-band amplitudes, $\omega_\pm = \Omega \pm \omega$, $\mathbf{k}_\pm = (q_x \pm k_x,\ \pm k_y)$.

It is assumed that $q_x \ll k_x$, which is typical for the existing theory of zonal-flow generation [5].

For the primary modes, starting equation (1) yields

$$\omega = \frac{\omega_R}{1 + r_R^2 \mathbf{k}_\perp^2}. \quad (6)$$

Here $\mathbf{k}_\perp^2 = k_x^2 + k_y^2$, $\omega_R \equiv k_y V_R$ is the frequency of the Rossby waves in the long-wavelength limit, $r_R^2\mathbf{k}_\perp^2 \ll 1$.

Taking the zonal-flow part of (1), one has (cf. eq. (17) of [5])

$$i\Omega\left(1 + r_R^2 q_x^2\right)\overline{h}_0 = f q_x^2 r_R^4 \left\langle \frac{\partial h}{\partial y}\frac{\partial h}{\partial x}\right\rangle. \quad (7)$$

Here $\langle\cdots\rangle$ is averaging over the small-scale oscillations. Allowing for eqs. (4) and (5), eq. (7) reduces to

$$i\Omega\overline{h}_0 = f\frac{q_x^2 r_R^4}{1 + r_R^2 q_x^2}\sum_{\mathbf{k}} k_y\left[2k_x\left(\widehat{h}_+\widetilde{h}_- + \widehat{h}_-\widetilde{h}_+\right) + \right.$$
$$\left. + q_x\left(\widehat{h}_+\widetilde{h}_- - \widehat{h}_-\widetilde{h}_+\right)\right]. \quad (8)$$

Now we take the side-band part of (1) arriving at

$$\widehat{h}_\pm = \pm k_y V_0 \widetilde{h}_\pm \frac{r_R^2\mathbf{k}_\perp^2}{\left(1 + r_R^2\mathbf{k}_{\perp\pm}^2\right) D_\pm}. \quad (9)$$

Here $V_0 = -i\left(gH_0/f\right) q_x \overline{h}_0$ is the zonal-flow part of the cross-field drift velocity, $\mathbf{k}_{\perp\pm}^2 = (q_x \pm k_x)^2 + k_y^2$,

$$D_\pm = \omega_\pm \mp \omega_R \big/ \left(1 + r_R^2\mathbf{k}_{\perp\pm}^2\right). \quad (10)$$

Using inequalities $\Omega \ll \omega$, $q_x \ll k_x$ and allowing for (6), we obtain

$$D_\pm = D_\pm^{(0)} + D_\pm^{(1)}, \quad (11)$$

where

$$D_\pm^{(0)} = \Omega - q_x V_g(\mathbf{k}), \quad (12)$$

$$D_\pm^{(1)} = \mp V_g' q_x^2 / 2. \quad (13)$$

Here $V_g(\mathbf{k})$ is the radial group velocity of the primary modes defined by

$$V_g(\mathbf{k}) = \frac{\partial\omega}{\partial k_x} = -\frac{2k_x\omega r_R^2}{1 + r_R^2\mathbf{k}_\perp^2}, \quad (14)$$

$V_g' \equiv \partial V_g / \partial k_x$ is its derivative, so that

$$V_g' = -\frac{2r_R^2\omega}{\left(1 + r_R^2\mathbf{k}_\perp^2\right)^2}\left(1 + r_R^2\mathbf{k}_\perp^2 - 4r_R^2 k_x^2\right). \quad (15)$$

Substituting (10) – (13) into (9), one has

$$\widehat{h}_\pm = \widehat{h}_\pm^{(0)} + \widehat{h}_\pm^{(1)}, \quad (16)$$

where

$$\widehat{h}_\pm^{(0)} = \pm\frac{k_y V_0 \widetilde{h}_\pm r_R^2\mathbf{k}_\perp^2}{\left(1 + r_R^2\mathbf{k}_\perp^2\right)\left(\Omega - q_x V_g\right)}, \quad (17)$$

$$\widehat{h}_\pm^{(1)} = \frac{k_y V_0 \widetilde{h}_\pm r_R^2\mathbf{k}_\perp^2}{\left(1 + r_R^2\mathbf{k}_\perp^2\right)\left(\Omega - q_x V_g\right)^2}\left[\frac{V_g' q_x^2}{2} - \frac{2r_R^2 k_x q_x D_\pm^{(0)}}{1 + r_R^2\mathbf{k}_\perp^2}\right]. \quad (18)$$

Substituting (16) – (18) into (8), we arrive at the zonal-flow dispersion relation

$$1 + \sum_{\mathbf{k}}\frac{F(\mathbf{k})}{\left[\Omega - q_x V_g(\mathbf{k})\right]^2} = 0, \quad (19)$$

where

$$F(\mathbf{k}) = gH_0\frac{q_x^4 r_R^4}{1 + r_R^2 q_x^2}\frac{V_g'}{\omega}k_y^2\mathbf{k}_\perp^2\left|\widetilde{h}_+\right|^2. \quad (20)$$

In contrast to (19), in [5] and [7] the case of single primary Rossby wave was considered. In this case (19) reduces to

$$\left(\Omega - q_x V_g\right)^2 = -F(\mathbf{k}_0), \quad (21)$$

where $\mathbf{k}_0$ is the wave vector of this mode. Transiting to the dimensionless form, one can see that (21) is the same as eq. (19) of [5] describing zonal-flow generation by the Rossby waves.

In [7] only the short-wave length limit of the Rossby waves and zonal flows was studied, i.e., the case $\left(r_R^2 q_x^2,\ r_R^2\mathbf{k}_\perp^2\right) \gg 1$. At the same time, the ratio $q_x/k_\perp$ was assumed to be finite. Therefore, in order to compare our zonal-flow dispersion relation with that of [7], one should take $r_R^2 q_x^2$ in (21) to be large and consider the limit $q_x/k_\perp \to 0$ in [7]. Then one obtains that (21) is the same as eq. (12) of [7].

For not too large k_x [5],

$$k_x^2 < k_y^3/3 + 1/r_R^2, \quad (22)$$

eq. (21) reduces to

$$1 + \frac{\Omega_0^2}{\left[\Omega - q_x V_g(\mathbf{k}_0)\right]^2} = 0, \quad (23)$$

where $\Omega_0^2 \equiv F(\mathbf{k}_0) > 0$. The roots of this equation are complex, $\mathrm{Im}\Omega \neq 0$. One of them has positive imaginary part, $\mathrm{Im}\Omega > 0$. Then one deals with the standard mechanism of zonal-flow generation similar to the negative mass instability of the linear theory [11] (the value Ω_0^2 plays the role of "minus of squared Langmuir frequency").

Turning to (15), one can see that the condition of this instability mechanism can be presented in the form

$$V_g' / \omega < 0. \tag{24}$$

It was noted in [5] and [7] that this condition is the same as the Lighthill criterion for modulation instability in nonlinear optics [12].

If the Lighthill instability criterion (24) is not satisfied,

$$V_g' / \omega > 0, \tag{25}$$

i.e., for

$$k_x^2 > k_y^2 / 3 + 1 / r_R^2 , \tag{26}$$

one has $F(\mathbf{k}_0) < 0$. Then, in the single-pump-wave case (19) leads to the zonal-flow dispersion relation of the form

$$1 - \frac{\Omega_0^2}{[\Omega - q_x V_g(\mathbf{k}_0)]^2} = 0, \tag{27}$$

where $\Omega_0^2 = -F(\mathbf{k}_0) > 0$. Turning to the linear theory of plasma instabilities [13], one can see that (27) is analogue of dispersion relation for a single cold beam. Its roots are real, $\mathrm{Im}\Omega = 0$. Meanwhile, it is well-known [13] that the stability properties of the system are radically changed if, instead of the single beam, one deals with two beams. Then the two-stream instability can take place.

Such an analogy leads to the idea that in the case of two pump waves, i.e., for

$$F(\mathbf{k}) = F(\mathbf{k}_1)\delta_{\mathbf{k}\mathbf{k}_1} + F(\mathbf{k}_2)\delta_{\mathbf{k}\mathbf{k}_2}, \tag{28}$$

a two-stream-like mechanism of zonal-flow generation can be revealed. In this case, instead of (27), one has the dispersion relation

$$1 - \frac{\Omega_1^2}{(\Omega - q_x V_{g1})^2} - \frac{\Omega_2^2}{(\Omega - q_x V_{g2})^2} = 0, \tag{29}$$

where $(\Omega_1^2,\ \Omega_2^2) = -[F(\mathbf{k}_1),\ F(\mathbf{k}_2)]$, $V_{gi} \equiv V_g(\mathbf{k}_i)$, $i = 1, 2$. Turning to [14], one can see that for not too large q_x, one of the roots of this dispersion relation has $\mathrm{Im}\Omega > 0$. Such a root describes the two-stream-like generation of zonal flows by the Rossby waves. More detailed analysis of the family of two-stream-like zonal-flow instabilities in magnetized plasma can be found in [8].

We have shown that, for two-humped spectra of the primary Rossby waves, a new, two-stream-like mechanism of zonal-flow generation can be revealed in the conditions when the Lighthill instability criterion is not satisfied. This results complements the picture of zonal-flow generation by the Rossby waves in the Earth's atmosphere predicted in [7].

In accordance with the ideas of [2] and [3] on similarity between wave properties of rotating shallow fluid and those of magnetized plasma, it seams to be reasonable to organize shallow-fluid experiments on simulation of the two-stream-like generation of zonal flows in magnetized plasma. At the same time, allowing for the ideas of [4] and [14], one can suggest that our results will favor further development of laboratory simulation of nonlinear phenomena in galaxies.

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