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Coulomb integrals in Liouville theory and Liouville gravity

V. A. Fateev[+*], *A. V. Litvinov*[+]

[+]*L.D.Landau Institute for Theoretical Physics RAS, 142432 Chernogolovka, Moscow reg. Russia*

[*]*Laboratoire de Physique Théorique et Astroparticules, Université Montpelier II, Pl.E. Bataillon, 34095 Montpelier, France*

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We study four-point correlation function in Liouville field theory. If one of the fields is degenerate, such a function is described in terms of Coulomb integrals. We find some non-trivial relations on these integrals, which can be used to obtain new exact results in conformal field theory. In particular, we calculate four-point correlation function in minimal quantum gravity. The result agrees with the results obtained recently by different methods [1, 2].

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Coulomb integrals play an important role in theoretical physics and mathematics. They were actively studied after 1984, when the modern approach to the two-dimensional conformal field theory (CFT) was formulated [3]. In particular, any multipoint correlation function in minimal models of CFT are expressed in terms of Coulomb integrals. Using these integrals, the structure constants in rational CFT describing the critical points of many interesting statistical systems [4, 5] and in $SU(2)$ WZW models [6] were calculated exactly. Also, they have application in the non-rational CFT like, for example, Liouville and Toda field theories and in perturbed CFT.

We start by recalling some basic facts about Liouville field theory (LFT)

$$\mathcal{L} = \frac{1}{4\pi}(\partial_a\varphi)^2 + \mu e^{2b\varphi}. \tag{1}$$

This theory has drawn a lot of attention mainly because it plays an important role in quantization of strings in non-critical dimension [7]. LFT is a conformal field theory with central charge c_L, parameterized in terms of coupling constant b as

$$c_L = 1 + 6Q^2, \tag{2}$$

with $Q = b + b^{-1}$ and μ is the scale parameter called the cosmological constant. Basic objects in this theory are the exponential fields

$$V_\alpha(z,\bar{z}) = e^{2\alpha\varphi}, \tag{3}$$

which are the primary fields of the Virasoro algebra with the conformal dimensions $\Delta_L(\alpha) = \alpha(Q-\alpha)$. Here z is complex coordinate on a plane. Below, for simplicity, we will neglect $\bar{z}$ dependence of the fields. The important property of LFT is that the fields V_α and $V_{Q-\alpha}$ have the same conformal dimension and really represent the same conformal field. It means, that they are related by a linear transformation

$$V_\alpha = R(\alpha)V_{Q-\alpha}, \tag{4}$$

with function

$$R(\alpha) = \frac{(\pi\mu\gamma(b^2))^{(Q-2\alpha)/b}}{b^2}\frac{\gamma(2b\alpha-b^2)}{\gamma(2-2\alpha/b+1/b^2)}, \tag{5}$$

which is known as the reflection amplitude, here and later $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. Relation (4) should be understood, as a set of identities on correlation functions.

It was noticed in [8], that any multipoint correlation function $\langle V_{\alpha_1}(x_1)\dots V_{\alpha_N}(x_N)\rangle$ exhibit a pole in the variable $\alpha = \sum \alpha_k$ if $\alpha = Q - nb$ with a residue being expressed in terms of the integral of correlation function of the free field exponents. Namely,

$$\operatorname*{res}_{\alpha=Q-nb}\langle V_{\alpha_1}(x_1)\dots V_{\alpha_N}(x_N)\rangle = \frac{(-\mu)^n}{n!}\times$$
$$\times\int\langle V_{\alpha_1}(x_1)\dots V_{\alpha_N}(x_N)\,V_b(t_1)\dots V_b(t_n)\rangle_0\,d^2\mathbf{t}, \tag{6}$$

here $\langle\dots\rangle_0$ denotes the correlation function of the exponential fields V_α, where φ now is free massless field and $d^2\mathbf{t} = \prod_k d^2t_k$. In the case of three points, integral in the r.h.s of (6) reads[1)]

$$I_n(\alpha_1,\alpha_2,\alpha_3) = \frac{(-\mu)^n}{n!}\times$$
$$\times\int\prod_k |t_k|^{-4b\alpha_1}|t_k-1|^{-4b\alpha_2}\mathcal{D}_n(t)^{-2b^2}\,d^2\mathbf{t}, \tag{7}$$

1)Using projective invariance, one can set $x_1 = 0$, $x_2 = 1$ and $x_3 = \infty$.

where

$$\mathcal{D}_n(t)=\prod_{i<j}^{n}|t_i-t_j|^2. \qquad (8)$$

Integral (7) was calculated exactly in [4]. Here we give the different derivation (simplest to our knowledge), based on the integral relation [9], which will be useful in the following

$$\frac{\pi^{-n}}{n!}\int\mathcal{D}_n(y)\prod_{i=1}^{n}\prod_{j=1}^{n+m+1}|y_i-t_j|^{2p_j}\,d^2\mathbf{y}=$$
$$=\frac{\prod_{j=1}^{n+1}\gamma(1+p_j)}{\gamma(1+n+\sum_j p_j)}\prod_{i<j}|t_i-t_j|^{2+2p_i+2p_j}\times$$
$$\times\frac{\pi^{-m}}{m!}\int\mathcal{D}_m(u)\prod_{i=1}^{m}\prod_{j=1}^{n+m+1}|u_i-t_j|^{-2-2p_j}\,d^2\mathbf{u}. \qquad (9)$$

Namely, one should apply identity (9) for $m=0$ from the right to the left and substitute

$$\mathcal{D}_n(t)^{-2b^2}=\mathcal{D}_n(t)\frac{\pi^{1-n}}{(n-1)!}\frac{\gamma(-nb^2)}{\gamma^n(-b^2)}\times$$
$$\times\int\mathcal{D}_{n-1}(y)\prod_{i=1}^{n-1}\prod_{j=1}^{n}|y_i-t_j|^{-2-2b^2}. \qquad (10)$$

After that, integral over t can be again calculated using Eq. (9). The remaining integral over y will be of the same type as (7), but with lower number of integrations. As result, we derive functional relation

$$I_n(\alpha_1,\alpha_2,\alpha_3)=$$
$$=\left(\frac{-\pi\mu}{\gamma(-b^2)}\right)\frac{\gamma(-nb^2)}{\gamma(2b\alpha_1)\gamma(2b\alpha_2)\gamma(2b\alpha_3+(n-1)b^2)}\times$$
$$\times I_{n-1}(\alpha_1+b/2,\alpha_2+b/2,\alpha_3). \qquad (11)$$

Repeating this procedure n times, we obtain that

$$I_n(\alpha_1,\alpha_2,\alpha_3)=\left(\frac{-\pi\mu}{\gamma(-b^2)}\right)^n\times$$
$$\times\prod_{j=0}^{n-1}\frac{\gamma(-(j+1)b^2)}{\gamma(2b\alpha_1+jb^2)\gamma(2b\alpha_2+jb^2)\gamma(2b\alpha_3+jb^2)}. \qquad (12)$$

Functional relation (11) can be used to continue $I_n(\alpha_1,\alpha_2,\alpha_3)$ to the non-integer n (the number of screenings V_b). Namely, one can substitute

$$n=(Q-\alpha_1-\alpha_2-\alpha_3)/b \qquad (13)$$

into Eq.(11) and consider it, as a functional relation for the three-point function $C(\alpha_1,\alpha_2,\alpha_3)$, which satisfies the condition that

$$\underset{\alpha=Q-nb}{\mathrm{res}}\,C(\alpha_1,\alpha_2,\alpha_3)\overset{\mathrm{def}}{=}I_n(\alpha_1,\alpha_2,\alpha_3). \qquad (14)$$

Functional relation (11) together with residue condition (14) allow us to find this quantity. An analytical expression for $C(\alpha_1,\alpha_2,\alpha_3)$ was proposed in [10, 11]

$$C(\alpha_1,\alpha_2,\alpha_3)=\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{(Q-\alpha)}{b}}\times$$
$$\times\frac{\Upsilon'(0)\prod_{k=1}^{3}\Upsilon(2\alpha_k)}{\Upsilon(\alpha-Q)\prod_{k=1}^{3}\Upsilon(\alpha-2\alpha_k)}, \qquad (15)$$

here $\Upsilon(x)$ is entire selfdual with respect to $b\to b^{-1}$ function satisfying functional relation

$$\Upsilon(x+b)=\gamma(bx)b^{1-2bx}\Upsilon(x) \qquad (16)$$

and defined by the integral

$$\log\Upsilon(x)=\int_0^\infty\frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2-\frac{\sinh^2\left(\frac{Q}{2}-x\right)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]. \qquad (17)$$

This function is symmetric $\Upsilon(x)=\Upsilon(Q-x)$ and has zeros in points

$$x=\begin{cases}-mb-nb^{-1},\\ Q+mb+nb^{-1},\end{cases}\qquad m,n=0,1,2,\ldots \qquad (18)$$

It is easy to see, that function $C(\alpha_1,\alpha_2,\alpha_3)$ satisfies relation (11), where n is defined by Eq. (13).

In the case of four points, integral in the r.h.s. of (6) has much more complicated analytical structure. The situation is simplified in the case, when one of the parameters $\alpha_k=-mb/2$ with $m=0,1,2,\ldots$ For these values of parameters, field $V_{-mb/2}(z)$ is degenerate at the level $m+1$ and satisfy differential equation of the same order in each variable z and $\bar{z}$. If the condition $-mb/2+\sum\alpha_k+nb=Q$ is satisfied, this correlation function possesses a pole with residue

$$J_m^{(n)}(\alpha_1,\alpha_2,\alpha_3|z)=\frac{(-\mu)^n}{n!}|z|^{2mb\alpha_1}|z-1|^{2mb\alpha_2}\times$$
$$\times\int\prod_{k=1}^{n}|t_k|^{-4b\alpha_1}|t_k-1|^{-4b\alpha_2}|t_k-z|^{2mb^2}\mathcal{D}_n^{-2b^2}(t)d^2\mathbf{t}, \qquad (19)$$

here n is the number of screening fields V_b, which appear in Eq. (6). One can reduce the number of integrations in Eq. (19) by multiple application of relation (9). Namely, it can be shown, that integral (19) satisfy the following remarkable property[2)]

$$|z|^{-2mb\alpha_1}|z-1|^{-2mb\alpha_2}J_m^{(n)}(\alpha_1,\alpha_2,\alpha_3|z)=$$
$$\Omega_m^n(\alpha_1,\alpha_2,\alpha_3)|z|^{-2nb\tilde{\alpha}_1}|z-1|^{-2nb\tilde{\alpha}_2}J_n^{(m)}(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3|z), \qquad (20)$$

[2)] For simplicity, we suppose, that $n>m$.

with $\tilde{\alpha}_k = \alpha_k + (n-m)b/2$ and

$$\Omega_m^n(\alpha_1,\alpha_2,\alpha_3) = \left(\frac{-\pi\mu}{\gamma(-b^2)}\right)^{n-m} \times$$
$$\times \prod_{j=0}^{n-m-1} \frac{\gamma(-(j+m+1)b^2)}{\gamma(2b\alpha_1 + jb^2)\gamma(2b\alpha_2 + jb^2)\gamma(2b\alpha_3 + jb^2)}. \quad (21)$$

Now, n appears in the integral $J_m^{(n)}(\alpha_1,\alpha_2,\alpha_3|z)$ as a parameter and we can easily perform continuation to the non-integer n. Continuation of $\Omega_m^n(\alpha_1,\alpha_2,\alpha_3)$ should be done in a such a way, that

$$\underset{\alpha=Q-nb+mb/2}{\mathrm{res}} \Omega_m(\alpha_1,\alpha_2,\alpha_3) = \frac{(-\mu)^m}{m!}\Omega_m^n(\alpha_1,\alpha_2,\alpha_3).$$

As a result,we obtain the following expression for the four-point function with one degenerate field in LFT

$$\langle V_{-\frac{mb}{2}}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle = \Omega_m(\alpha_1,\alpha_2,\alpha_3) \times$$
$$\times |z|^{2mb\alpha_1}|z-1|^{2mb\alpha_2}\,\mathbf{J}_m(A,B,C|z), \quad (22)$$

where function $\mathbf{J}_m(A,B,C|z)$ defined by

$$\mathbf{J}_m(A,B,C|z) =$$
$$= \int \mathcal{D}_m^{-2b^2}(t) \prod_{k=1}^m |t_k|^{2A}|t_k-1|^{2B}|t_k-z|^{2C}\, d^2\mathbf{t}, \quad (23)$$

with

$$A = b\,(\alpha - 2\alpha_1 - Q + mb/2)\,,$$
$$B = b\,(\alpha - 2\alpha_2 - Q + mb/2)\,,$$
$$C = b\,(Q + mb/2 - \alpha)\,,$$

and normalization constant $\Omega_m(\alpha_1,\alpha_2,\alpha_3)$ is

$$\Omega_m(\alpha_1,\alpha_2,\alpha_3) = \frac{(-\mu)^m}{m!}\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{(Q-\alpha-mb/2)}{b}} \times$$
$$\times \frac{\Upsilon'(-mb)\prod_{k=1}^3 \Upsilon(2\alpha_k)}{\Upsilon(\alpha - Q - \frac{mb}{2})\prod_{k=1}^3 \Upsilon(\alpha - 2\alpha_k + \frac{mb}{2})}, \quad (24)$$

here $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

Up to the normalization constant $\Omega_m^n(\alpha_1,\alpha_2,\alpha_3)$ the expression for this integral was derived in [12] from the condition, that this function satisfies $m+1$-order differential equation. Function $\mathbf{J}_m(A,B,C|z)$, defined by Eq. (23), possesses many remarkable properties. Some of them can be derived from the reflection relation (4) applied to the one of the fields V_α:

- $\hat{\sigma}^{(\infty)}$ – relation (reflection $\alpha_3 \to Q - \alpha_3$),

$$\mathbf{J}_m(A,B,C|z) = G_m(A,B,C)\,\mathbf{J}_m(A',B',C'|z), \quad (25a)$$

with

$$A' = -1 - B + (m-1)b^2,\ B' = -1 - A + (m-1)b^2,$$
$$C' = 1 + A + B + C - (m-1)b^2.$$

- $\hat{\sigma}^{(0)}$ – relation (reflection $\alpha_1 \to Q - \alpha_1$),

$$\mathbf{J}_m(A,B,C|z) = G_m(A,B,C)\,|z|^{2m(C'-B)} \times$$
$$\times \mathbf{J}_m(A+B-C',C',B'|z), \quad (25b)$$

- $\hat{\sigma}^{(1)}$ – relation (reflection $\alpha_2 \to Q - \alpha_2$),

$$\mathbf{J}_m(A,B,C|z) = G_m(A,B,C)\,|z-1|^{2m(C'-A)} \times$$
$$\times \mathbf{J}_m(C',A+B-C',A'|z). \quad (25c)$$

Constant $G_m(A,B,C)$ is the same in all relations (25)

$$G_m(A,B,C) =$$
$$= \prod_{j=0}^{m-1} \frac{\gamma(1+A-jb^2)\gamma(1+B-jb^2)\gamma(1+C-jb^2)}{\gamma(2+A+B+C-(m-1+j)b^2)}.$$

Relations (25) are useful for analytical continuation and for calculation of different Coulomb integrals.

Now we apply Coulomb integrals for the calculation for the four-point correlation function in the minimal Liouville gravity, which is described by generalized minimal model (GMM) [1] of CFT with central charge $C_M = 1 - 6\left(b - b^{-1}\right)^2$, coupled to the LFT in such a way, that $c_L + c_M = 26$. GMM theory includes a continuous number of primary fields Φ_α with conformal dimension $\Delta_M(\alpha) = \alpha(\alpha + b - b^{-1})$[3]. These models were studied in [1], where more detailed description of GMM as well as Liouville theory can be found. Here we choose normalization of the fields Φ_α in a such a way, that

$$\langle \Phi_{\alpha-b}(z)\Phi_{\alpha-b}(0)\rangle = \mathcal{N}^{-2}(\alpha)\,|z|^{-4\Delta_M(\alpha-b)}, \quad (26)$$

$$\mathcal{N}(\alpha) = (\pi\mu\gamma(b^2))^{-\frac{\alpha}{b}}\left[\gamma(2\alpha b - b^2)\gamma(2\alpha b^{-1} - b^{-2})\right]^{\frac{1}{2}}.$$

The main problem in minimal Liouville gravity is to evaluate correlation function of operators $U_\alpha = \Phi_{\alpha-b}V_\alpha$, which have conformal dimension $\Delta_L(\alpha) + \Delta_M(\alpha - b) = 1$ and hence $(1,1)$ form $U_\alpha(z)d^2z$ can be integrated in an invariant way. Integrated N-point correlation functions are invariant objects, which depend only on parameters α_k. Due to the group of diffeomorphisms $SL(2,C)$, which is the symmetry of the theory, the number of integrations in N-point correlation function can be reduced

[3] We note, that central charge C_M and conformal dimension $\Delta_M(\alpha)$ can be derived from the corresponding values C_L and $\Delta_L(\alpha)$ by the substitution $b \to -ib$ and $\alpha \to i\alpha$.

to $N-3$. Namely, we can fix the coordinates of any three fields at the points 0, 1 and ∞. This fact is well known in string dual models, where similar integrals appear in the tree string amplitudes. Three-point correlation function of the fields U_{α_k} does not contain integration and is rather simple [1]:

$$\langle U_{\alpha_1}(0)U_{\alpha_2}(1)U_{\alpha_3}(\infty)\rangle = \left(\gamma(b^2)\gamma(b^{-2}-1)\right)^{-\frac{1}{2}}\hat{\Omega}, \tag{27}$$

here

$$\hat{\Omega} = b^{-1}\left[\pi\mu\gamma(b^2)\right]^{\frac{Q}{b}}\gamma(b^2)\gamma(b^{-2}-1). \tag{28}$$

Four-point function will have one integration. We define (following Ref. [1, 2]) four-point function $\Sigma_m(\alpha_1,\alpha_2,\alpha_3)$, which contains one matter field $\Phi_{mb/2}$, degenerate at the level $m+1$

$$\Sigma_m(\alpha_1,\alpha_2,\alpha_3) = = \int d^2z\,\langle U_{\frac{(m+2)b}{2}}(z)U_{\alpha_1}(0)U_{\alpha_2}(1)U_{\alpha_3}(\infty)\rangle. \tag{29}$$

Corresponding matter four-point function satisfies differential equation of the order $m+1$ and can be also represented by the m dimensional integral similar to the Liouville case (23)

$$\begin{aligned}&\langle\Phi_{\frac{mb}{2}}(z)\Phi_{\alpha_1-b}(0)\Phi_{\alpha_2-b}(1)\Phi_{\alpha_3-b}(\infty)\rangle = \\ &=\Lambda_m(\alpha_1,\alpha_2,\alpha_3)|z|^{2mb(\alpha_1-b)}|z-1|^{2mb(\alpha_2-b)}\times \\ &\times\int d^2\mathbf{t}\prod_{i=1}^{m}|t_i|^{2\tilde{A}}|t_i-1|^{2\tilde{B}}|t_i-z|^{2\tilde{C}}\mathcal{D}_m(t)^{2b^2},\end{aligned} \tag{30}$$

where

$$\begin{aligned}\tilde{A} &= b\left(\alpha-2\alpha_1-Q-(m-2)b/2\right),\\ \tilde{B} &= b\left(\alpha-2\alpha_2-Q-(m-2)b/2\right),\\ \tilde{C} &= -b\left(\alpha-Q+(m-2)b/2\right),\end{aligned} \tag{31}$$

and

$$\begin{aligned}\Lambda_m &= \frac{\mu^{-m}}{\pi^{2m}m!}\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{(\alpha-Q+3mb/2+b)}{b}}\times\\ &\times\frac{\Upsilon(\alpha-Q+\frac{mb}{2})\prod_{k=1}^{3}\Upsilon(\alpha-2\alpha_k-\frac{mb}{2})}{\Upsilon((m+2)b)\prod_{k=1}^{3}\Upsilon(2\alpha_k)},\end{aligned} \tag{32}$$

here $\alpha=\alpha_1+\alpha_2+\alpha_3$.

Integral for four-point function in GMM with one degenerate field $\Phi_{mb/2}$ and three arbitrary fields Φ_{α_k} can be obtained by substitution $\alpha\to i\alpha$ and $b\to -ib$ into integral for four-point function in the Liouville theory (23), while normalization factor $\Lambda_m(\alpha_1,\alpha_2,\alpha_3)$ can be derived, following the lines of calculation of $\Omega_m(\alpha_1,\alpha_2,\alpha_3)$ and taking into account normalization condition (26). One also has to remember, that variables α_k in Eq. (30) are shifted: $\alpha_k\to\alpha_k-b$.

Four-point correlation function of the corresponding Liouville fields

$$\langle V_{(m+2)b/2}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle \tag{33}$$

is more complicated. It simplifies, however, when

$$\epsilon = (m+2)b/2+\alpha+nb-Q\to 0. \tag{34}$$

for this values of parameters, matter four-point function (30) exhibits a zero. In the limit $\varepsilon\to 0$ first Υ function in (32) has asymptotic

$$\Upsilon(\alpha-Q+\frac{mb}{2})\to\frac{b^{-n-(n+1)(n+2)b^2}}{\prod_{j=1}^{n+1}\gamma(-jb^2)}\frac{\Upsilon(b)}{\Gamma(\epsilon)}, \tag{35}$$

while Liouville four-point function (33) possesses the asymptotic, which can be expressed in terms of n-dimensional integral (23):

$$\begin{aligned}&\langle V_{\frac{(m+2)b}{2}}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle\to\\ &\to\Gamma(\epsilon)\frac{(-\mu)^n}{n!}|z|^{-2b\alpha_1(m+2)}|z-1|^{-2b\alpha_2(m+2)}\times\\ &\times\mathbf{J}_n(-2b\alpha_1,-2b\alpha_2,-(m+2)b^2|z).\end{aligned} \tag{36}$$

Multiplying (30) and (36), performing limit $\epsilon\to 0$ and integrating over z, we obtain that

$$\begin{aligned}&\Sigma_m(\alpha_1,\alpha_2,\alpha_3) = -\frac{\hat{\Omega}}{\pi^{n+m}m!n!}\times\\ &\times\prod_{j=0}^{n+m}\gamma(2b\alpha_1+jb^2)\gamma(2b\alpha_2+jb^2)\gamma(2b\alpha_3+jb^2)\times\\ &\times\prod_{j=1}^{m+1}\left(\frac{\gamma(b^2)}{\gamma(jb^2)}\right)\prod_{j=1}^{n+1}\left(\frac{\gamma(-b^2)}{\gamma(-jb^2)}\right)\mathbf{J}_{nm}(\alpha_1,\alpha_2,\alpha_3),\end{aligned} \tag{37}$$

where

$$\begin{aligned}&\mathbf{J}_{nm}(\alpha_1,\alpha_2,\alpha_3)=\int d^2z\,d^2\mathbf{t}\,d^2\mathbf{s}\,\mathcal{D}_m(t)^{2b^2}\mathcal{D}_n(s)^{-2b^2} =\\ &=|z|^{-2mb^2-4b\alpha_1}|z-1|^{-2mb^2-4b\alpha_2}\times\\ &\times\prod_{i=1}^{m}|t_i|^{-2(n+m)b^2-4b\alpha_1}|t_i-1|^{-2(n+m)b^2-4b\alpha_2}|t_i-z|^{2(n+2)b^2}\\ &\times\prod_{i=1}^{n}|s_i|^{-4b\alpha_1}|s_i-1|^{-4b\alpha_2}|s_i-z|^{-2(m+2)b^2}.\end{aligned} \tag{38}$$

The integral $\mathbf{J}_{nm}(\alpha_1,\alpha_2,\alpha_3)$ can be calculated exactly. It is useful to apply relation (25a) to the n-fold integral over variables s and similar relation to the m-fold integral over variables t (in the last case, with substitution $b^2 \to -b^2$), and rewrite $\mathbf{J}_{nm}(\alpha_1, \alpha_2, \alpha_3)$ as:

$$\mathbf{J}_{nm}(\alpha_1,\alpha_2,\alpha_3) = \prod_{j=1}^{m+1}\gamma(jb^2)\prod_{j=1}^{n+1}\gamma(-jb^2)\times \times\frac{\gamma(2b\alpha_1+nb^2)\gamma(2b\alpha_2+nb^2)\gamma(2b\alpha_3+nb^2)}{\prod_{j=1}^{m+n+1}\gamma(jb^2)\gamma(-jb^2)}\times \times\prod_{j=0}^{n+m}\left[\gamma(2b\alpha_1+jb^2)\gamma(2b\alpha_2+jb^2)\gamma(2b\alpha_3+jb^2)\right]^{-1}\times \times H_{nm}(\alpha_1+nb^2/2,\alpha_2+nb^2/2,\alpha_3+nb^2/2), \quad (39)$$

here $H_{nm}(a_1, a_2, a_3)$ is the integral, which is evidently symmetric under the substitution $n \leftrightarrow m$ and $b^2 \to -b^2$

$$H_{nm}(a_1,a_2,a_3) = \int d^2z\, d^2\mathbf{s}\, d^2\mathbf{t}\ \mathcal{D}_m(t)^{2b^2}\mathcal{D}_n(s)^{-2b^2}\times \times|z|^{-4ba_1+2(n-m)b^2}|z-1|^{-4ba_2+2(n-m)b^2}\times \times\prod_{i=1}^{m}|t_i|^{-2+4ba_2+2b^2}|t_i-1|^{-2+4ba_1+2b^2}|t_i-z|^{-2+4ba_3+2b^2}\times \times\prod_{i=1}^{n}|s_i|^{-2+4ba_2-2b^2}|s_i-1|^{-2+4ba_1-2b^2}|s_i-z|^{-2+4ba_3-2b^2}, \text{here}\quad a_1+a_2+a_3 = b^{-1}-(n-m)b. \quad (40)$$

Integral (40), despite its complex form, has very simple analytical structure in parameters a_k. Namely, it has only simple poles in the points $2ba_k = 1, 2, \ldots$ and symmetric with respect to $a_i \leftrightarrow a_j$. These properties permit to calculate it exactly with the result

$$H_{nm}(a_1,a_2,a_3) = \pi^{m+n+1}(m+1)!\,(n+1)!\times \times\frac{\prod_{j=1}^{m+n+1}\gamma(jb^2)\gamma(-jb^2)}{\gamma(-b^2)^n\gamma(b^2)^m}\frac{1}{\gamma(2ba_1)\gamma(2ba_2)\gamma(2ba_3)}. \quad (41)$$

Using Eq. (39), we obtain the following expression for the integral

$$\mathbf{J}_{nm}(\alpha_1,\alpha_2,\alpha_3) = \pi^{n+m+1}\times \times(m+1)!\,(n+1)!\prod_{j=1}^{m+1}\left(\frac{\gamma(jb^2)}{\gamma(b^2)}\right)\prod_{j=1}^{n+1}\left(\frac{\gamma(-jb^2)}{\gamma(-b^2)}\right)\times \times\prod_{j=0}^{n+m}\left[\gamma(2b\alpha_1+jb^2)\gamma(2b\alpha_2+jb^2)\gamma(2b\alpha_3+jb^2)\right]^{-1}. \quad (42)$$

Taking into account Eqs. (42) and (37), we obtain the final expression for the four-point function (37):

$$\Sigma_m(\alpha_1,\alpha_2,\alpha_3) = -\pi\hat{\Omega}\,(1+n)(1+m). \quad (43)$$

Now we substitute $n+1 = (Q-\alpha-mb/2)/b$, with $\alpha = \sum_k \alpha_k$, in Eq. (43). As a result, we obtain

$$\Sigma_m(\alpha_1,\alpha_2,\alpha_3) = \pi b^{-1}\hat{\Omega}\,(1+m)\big(\alpha+mb/2-Q\big). \quad (44)$$

The answer (44) is in complete agreement with the results of Ref. [1, 2] in the domain of convergency of the integral $\mathbf{J}_{nm}(\alpha_1, \alpha_2, \alpha_3)$:

$$\alpha-2\alpha_k > mb/2,\ \ \alpha_i < Q/2-mb/2. \quad (45)$$

We note in the conclusion, that using our method, we can consider also the dual case, where one of the Liouville fields is degenerate at the level $m+1$ and hence Liouville four-point function is given by Eqs. (22)–(24); while the screening condition $\alpha - mb/2 = Q + (n+2)b$ is satisfied for the matter correlation function, which due to this condition can be expressed through n-dimensional integral. In this case, all calculations can be done in the same way. The integral for the dual four-point function

$$\tilde{\Sigma}_m(\alpha_1,\alpha_2,\alpha_3) = = \int\langle U_{-mb/2}(z)U_{\alpha_1}(0)U_{\alpha_2}(1)U_{\alpha_3}(\infty)\rangle d^2z$$

is convergent in the domain:

$$\alpha-2\alpha_k < Q-mb/2,\ \ \alpha_i > (m+1)b/2, \quad (46)$$

and can be calculated exactly. The result, which can be expressed in terms of function $\Sigma_m(\alpha_1, \alpha_2, \alpha_3)$, defined by Eq. (44), with the substitution

$$\tilde{\Sigma}_m(\alpha_1,\alpha_2,\alpha_3) = \Sigma_{-m-2}(\alpha_1,\alpha_2,\alpha_3). \quad (47)$$

We suppose to study four-point correlation functions $\Sigma_m(\alpha_1, \alpha_2, \alpha_3)$ and $\tilde{\Sigma}_m(\alpha_1, \alpha_2, \alpha_3)$ outside the domains (45) and (46) in the future publication.

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