Kelvin-wave turbulence generated by vortex reconnections

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Reconnections of quantum vortex filaments create sharp bends which degenerate into propagating Kelvin waves. These waves cascade their energy down-scale and their waveaction up-scale via weakly nonlinear interactions, and this is the main mechanism of turbulence at the scales less than the inter-vortex distance. In case of an idealised forcing concentrated around a single scale k_0, the turbulence spectrum exponent has a pure direct cascade form $-17/5$ at scales $k > k_0$ [2] and a pure inverse cascade form -3 at $k < k_0$ [9]. However, forcing produced by the reconnections contains a broad range of Fourier modes. What scaling should one expect in this case? In this Letter I obtain an answer to this question using the differential model for the Kelvin wave turbulence introduced in [8]. The main result is that the direct cascade scaling dominates, i.e. the reconnection forcing is more or less equivalent to a low-frequency forcing.

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1. Differential equation model for Kelvin wave turbulence.
Superfluid turbulence, when excited at scales much greater than the mean separation between quantum vortices, behaves similarly to turbulence in classical fluids at such large scales in that it develops a Richardson-like cascade characterised by Kolmogorov spectrum [1]. However, quantum turbulence starts feeling discreteness when the energy cascade reaches down to the length-scales comparable to the mean inter-vortex separation distance. In superfluids near zero temperature, there is no normal component and, therefore, there is no viscid of frictional dissipation in the system. Even though part of the turbulent energy is lost to sound radiation during the vortex reconnection processes, the major part of it is believed to be continuing to cascade to the scales below the inter-vortex separation scale via nonlinear interactions of Kelvin waves [2–7]. Following [8] I will refer to this state characterised by random non-linearly interacting Kelvin waves as "kelvulence" (i.e. Kelvin turbulence). Kozik and Svistunov [5] used the weak turbulence approach to kelvulence and derived a six-wave kinetic equation (KE) for the spectrum of weakly nonlinear Kelvin waves. Based on KE, they derived a spectrum of waveaction that corresponds to the constant Kolmogorov-like cascade of energy from small to large wavenumbers,

$$n_k \sim k^{-17/5}. \tag{1}$$

Because the number of waves in the leading resonant process is even (i.e. 6), KE conserves not only the total energy but also the total waveaction of the system. The systems with two positive conserved quantities are known in turbulence to possess a dual cascade behaviour. For the Kelvin waves, besides the direct energy cascade there also exists an inverse cascade of waveaction [9],

$$n_k \sim k^{-3}. \tag{2}$$

Numerical confirmation of the direct cascade spectrum (1) was given by Kozik and Svistunov [7] who forced the system at the largest scales. To date, there has been no simulations with forcing concentrated at the smallest scales and, therefore, there is no numerical confirmation of the inverse cascade spectrum.

On the other hand, in superfluids kelvulence is generated by vortex reconnections which is not concentrated in either large or small scales, but it has a continuous k-space distribution. Indeed, a sharp bend on the vortex line produced by a reconnection has spectrum $n_k \sim k^{-4}$. What scaling should we expect in kelvulence pumped by the reconnections, – forward cascade, inverse cascade or a mixture of thereof? In the present paper I will answer this question using a differential approximation model (DAM) introduced in [8],

$$\dot{n} = \frac{C}{\kappa^{10}} \omega^{1/2} \frac{\partial^2}{\partial \omega^2} \left(n^6 \omega^{21/2} \frac{\partial^2}{\partial \omega^2} \frac{1}{n} \right) + F_k - D_k, \tag{3}$$

where κ is the vortex line circulation, C is a dimensionless constant and $\omega = \omega(k) = \frac{\kappa}{4\pi} k^2$ is the Kelvin wave frequency (we ignore logarithmic factors). Here F_k and D_k are the terms describing forcing and dissipation of Kelvin waves.

In absence of forcing and dissipation, $F_k = D_k = 0$, DAM preserves the energy

$$E = \int \omega^{1/2} n \, d\omega \tag{4}$$

and the waveaction

$$N = \int \omega^{-1/2} n \, d\omega. \tag{5}$$

In this case equation (3) has both the direct cascade solution (1) and the inverse cascade solution (2). It also has thermodynamic Rayleigh-Jeans solutions,

$$n = T/(\omega + \mu). \tag{6}$$

where T and μ are constants having a meaning of temperature and the chemical potential respectively.

Now let us assume that the forcing is due to the vortex reconnections so that

$$F_k = \lambda \omega^{-2}, \tag{7}$$

where λ is a constant proportional to the mean frequency of reconnections. For now let us ignore the dissipation by putting $D_k = 0$. Dissipation of kelvulence is due to either sound emission [8] or due to a friction with the normal component [10]. This dissipation acts at very short scales and I will discuss its role in the end of this paper.

2. Directions of the energy and waveaction cascades. First of all, it is instructive to study directions of the energy and the waveaction cascades. For this, let us re-write equation (3) in two different forms: a continuity equation for the waveaction,

$$\dot{n} = -\partial_k \eta = -2\omega^{1/2} \partial_\omega \eta, \tag{8}$$

and a continuity equation for the energy

$$\omega \dot{n} = -\partial_k \epsilon = -2\omega^{1/2} \partial_\omega \epsilon, \tag{9}$$

where η and ϵ are the spectral fluxes of the waveaction and of the energy respectively,

$$\eta = -\frac{C}{2\kappa^{10}} \partial_\omega R, \tag{10}$$

and

$$\epsilon = \frac{C}{2\kappa^{10}}(R - \omega \partial_\omega R), \tag{11}$$

with

$$R = n^6 \omega^{21/2} \frac{\partial^2}{\partial \omega^2} \frac{1}{n}. \tag{12}$$

Note that $\eta = 0$ and $\epsilon =$const on the direct cascade solution (1) and, respectively, $\eta =$const and $\epsilon = 0$ on the inverse cascade solution (2). More generally, on power-law spectra $n_k = k^\nu$ we have $\eta > 0$ for $-\infty < \nu < -17/5$ and for $-1 < \nu < 0$ (and $\eta \le 0$ otherwise), whereas

$\epsilon > 0$ for $-\infty < \nu < -3$ and for $-1 < \nu < 0$ (and $\eta \le 0$ otherwise). Particularly, if we take spectrum of a sharp reconnection-produced bend, $n_k = k^{-4}$, then both energy and waveaction cascades are direct, $\eta > 0$ and $\epsilon > 0$. This fact is an indication that kelvulence forced by reconnections should be dominated by the direct cascade rather than the inverse cascade scalings. However, the steady state spectrum will be different from $n_k = k^{-4}$ due to the redistributions of the waveaction and the energy by the nonlinear wave interactions. Below, we will study such a steady state using a reduced version of DAM.

3. Reduced DAM for Kelvulence forced by reconnections. In principle, one can study steady states on kelvulence forced by reconnections using DAM as given by equation (3). However, it is impossible to find a general steady state analytical solution in this case and one needs to resort to numerics. On the other hand, most essential details and a full analytical treatment is possible using a reduced version of DAM,

$$\dot{n} = \frac{C'}{\kappa^{10}} \omega^{-1/2} \partial_\omega \left(n^4 \omega^8 \partial_\omega \left(n \omega^{3/2} \right) \right) + \lambda \omega^{-2}, \tag{13}$$

where C' is an order-one constant. In this version (for $\lambda = 0$) DAM also conserves both the energy and the waveaction and describes their respective cascade states (1) and (2), but it no longer has thermodynamic Rayleigh-Jeans solutions (6). Note that the energy and the waveaction fluxes in this model are respectively

$$\epsilon = -\frac{C'}{2\kappa^{10}} n^4 \omega^8 \partial_\omega (n \omega^{3/2}), \tag{14}$$

and

$$\eta = -\frac{C'}{2\kappa^{10}} n^4 \omega^{34/5} \partial_\omega (n \omega^{17/10}). \tag{15}$$

Integrating equation (13) once, we get

$$\epsilon = \epsilon_0 - \lambda \omega^{-1/2}, \tag{16}$$

where ϵ_0 is a (positive) constant having a meaning of the asymptotic value of the energy flux at large frequencies. Integrating one more time we get

$$n = \left(\frac{10}{C'}\right)^{1/5} \kappa^2 \omega^{-3/2} \left(\epsilon_0 \omega^{-1} - \frac{2\lambda}{3}\omega^{-3/2} - \eta_0\right)^{1/5}, \tag{17}$$

where η_0 is a (negative) constant having a meaning of the asymptotic value of the wavenumber flux at large frequencies. If there is no additional (with respect to the reconnections) forcing then $\eta_0 = 0$, so that

$$n = \left(\frac{10}{C'}\right)^{1/5} \kappa^2 \omega^{-17/10} \left(\epsilon_0 - \frac{2\lambda}{3}\omega^{-1/2}\right)^{1/5}. \tag{18}$$

At large frequencies, this solution asymptotically approaches to the direct cascade scaling (1). The inverse cascade scaling (2) does not form in any frequency range. One can relate the asymptotic value of the energy flux to the minimal frequency of the system $\omega_{\min} = \kappa k_{\min}^2 = \kappa(2\pi/L)^2$, where L is the length of the vortex filament. This relation follows from (16) and a the condition that the flux ϵ is zero at $\omega_{\min}$, i.e.

$$\epsilon_0 = \lambda \omega_{min}^{-1/2}. \tag{19}$$

At the minimal frequency, the spectrum tends to a finite value

$$n(\omega_{min}) = (10\lambda/3C')^{1/5} \kappa^2 \omega_{min}^{-9/5}. \tag{20}$$

The spectrum (18) is less steep near $\omega_{\min}$ than in the free-cascade range at large ω, but the slope remains negative for all ω (i.e. there is no maximum).

4. Discussion. In this paper, I studied the Kelvin wave turbulence (kelvulence) generated by the vortex reconnections and evolving due to nonlinear wave interactions. For this, I used the differential approximation model (DAM) of kelvulence previously introduced in [8] and its reduced version (13). The stationary solution of this model (18) describes a state in which the energy flux is directed toward higher frequencies and it grows from zero at a minimal frequency $\omega_{\min}$ to a constant asymptotic value (19) at large frequencies. In this asymptotic range, the spectrum has a pure direct cascade scaling (1). Thus, the answer to the question asked in the beginning of this paper is that it is the direct rather than the inverse cascade scaling that dominates in kelvulence excited by reconnections. However, a certain amount of waveaction is also produced by the reconnections per unit time near $\omega_{\min}$; it leaks to smaller frequencies and must be absorbed at the $\omega_{\min}$ boundary (otherwise there would be a pile-up of spectrum near $\omega_{\min}$ without reaching a steady state). This absorption seems to arise naturally in the system because the waveaction conservation takes place only in the weak turbulence regime which breaks down near $\omega_{\min}$, particularly due to the mode discreteness.

So far we neglected dissipation the role of which is to absorb the energy cascade at very high frequencies. In superfluids there is no viscosity and the dissipation is due to either sound generation by short Kelvin waves (near absolute zero temperature) or due to a friction

with the normal component (at higher temperatures) [11]. Study of the dissipation effects on the direct cascade in kelvulence within DAM approach was done in [8] and [10]. It was shown that both of these dissipation mechanisms do not affect the direct cascade spectrum at low ω but they arrest at some large frequency which results in a sharp cut-off of the spectrum at some maximum frequency $\omega_{\max}$. A finite cut-off due to phonon radiation was earlier predicted also in [12] and [13].

Assuming that $\omega_{\max} \gg \omega_{\min}$, we see that the reconnection forcing and the radiative/frictional dissipation are separated in the frequency space: the forcing is effectively concentrated near $\omega_{\min}$ and the dissipation acts only near $\omega_{\max}$. This justifies the approach taken in this paper where we neglected the radiational and frictional radiation while considering the effect of the reconnection forcing.

An interesting problem for future studies would be numerical simulation of DAM given by the fourth-order equation (3) including the reconnection forcing as well as the radiation or/and friction dissipation, and comparison of it with direct numerical simulations of the vortex line with the same forcing and dissipation mechanism. It would also be interesting to establish what effects on kelvulence have curved geometry and non-stationarity of the vortex filaments.

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