

Modulational instability of nonlinearly interacting incoherent sea states

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The modulational instability of nonlinearly interacting spatially incoherent Stokes waves is analyzed. Starting from a pair of nonlinear Schrödinger equations, we derive a coupled set of wave-kinetic equations by using the Wigner transform technique. It is shown that the partial coherence of the interacting waves induces novel effects on the dynamics of crossing sea states.

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Extremely large amplitude freak waves (also referred to as rogue or giant waves) in oceans are well-known [1–4]. Freak waves, which are steep waves and which can appear quickly from a relatively calm sea, are responsible for the loss of many human lives and ships. Therefore, it is important to understand the nonlinear propagation [5–9] of such waves in dispersive fluids that are far from a stationary state. Specifically, we recall that in non-stationary oceans, the dynamics of modulated Stokes waves is under certain circumstances governed by a nonlinear Schrödinger (NLS) equation [10], from which so called freak waves can occur. We note that the occurrence of freak waves may also be due to, e.g. a linear superposition of harmonic waves or wave-current interactions, and can thus be described by a different set of equations, such as the Zakharov equation [11]. The latter depicts a modulational instability (also known as the Benjamin-Feir instability [12, 13] in fluid mechanics) of a constant amplitude carrier wave, and indicates that modulationally unstable waves can form localized envelope disturbances [14]. In Refs. [15] and [16] the effects of a random sea swell on the stability was investigated using the single sea state nonlinear deep water wave picture.

Recently, Onorato et al. [17] developed a two-dimensional weakly nonlinear model for two coherent

Stokes waves in deep water with two different directions of propagation. For one-dimensional wave propagation, Ref. [17] presented specific results for the modulational instability of two nonlinearly coupled coherent Stokes waves. It was found that freak waves may occur as a result of the modulational instability, and the growth rate for the crossing sea state modulational instability was found to be larger than that for the single sea state case. Since in reality the waves are incoherent it is however appropriate to examine the effects of random phases using perturbation theory. The presence of a spatial random phase of the background sea state gives rise to spectral broadening of the wave distribution function (as compared to the usual modulational perturbations around a monochromatic background wave distribution), and the modulational instability of nonlinearly coupled Stokes waves can be affected accordingly. This is the objective of the present Letter.

In the following, we thus analyze the statistical properties of one-dimensional incoherent crossing sea states. In particular, we investigate the modulational instability properties of these nonlinearly interacting waves. A generalized distribution function, which is valid for partially coherent waves, will be obtained. Using perturbation theory, it will be shown that partial coherence in terms of a random phase approximation gives rise to spectral broadening of the background crossing sea states. This broadening tends to stabilize the inherent system modulational instability, enabling the interaction between the waves over longer distances before any perturbation grows to appreciable levels forming freak waves. In fact, for a wide enough spectral distribution of waves the modulational instability can be suppressed.

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In Ref. [17] the equations

$$\frac{\partial A}{\partial t} - i\alpha \frac{\partial^2 A}{\partial x^2} + i(\xi|A|^2 + 2\zeta|B|^2)A = 0 \quad (1)$$

and

$$\frac{\partial B}{\partial t} - i\alpha \frac{\partial^2 B}{\partial x^2} + i(\xi|B|^2 + 2\zeta|A|^2)B = 0, \quad (2)$$

for one-dimensional comoving crossing sea states were presented. Here A and B represents the surface elevation of the crossing sea states, α is the group velocity dispersion parameter, ξ is the nonlinear self-interaction parameter, and ζ determines the strength of the nonlinear interaction between the crossing sea states. We note that the sign of α can always be chosen positive by a change of time coordinate. Thus, we take $\alpha > 0$, while the nonlinear parameters may be positive as well as negative. It can also be noted that the one-dimensional nonlinear Schrödinger equation has been shown to be in excellent agreement with laboratory experiments [18].

There are traveling pure Stokes wave solutions $A = A_0 \exp(-i\omega_A t)$ and $B = B_0 \exp(-i\omega_B t)$ of Eqs. (1) and (2), where $\omega_A = \xi A_0^2 + 2\zeta B_0^2$ and $\omega_B = \xi B_0^2 + 2\zeta A_0^2$. Letting $A(t, x) = A_0[1 + a(t, x)] \exp(-i\omega_A t)$ and $B(t, x) = B_0[1 + b(t, x)] \exp(-i\omega_B t)$, where $a, b \ll 1$, we linearize Eqs. (1) and (2), take the real and imaginary parts of the resultant equations, and Fourier analyze them against the frequency Ω and wavenumber K of the modulations. The resulting dispersion relation is [17]

$$\Omega^4 - 2\alpha K^2(\alpha K^2 + \xi A_0^2 + \xi B_0^2)\Omega^2 + \alpha^2 K^4 \times \\ \times [(\alpha K^2 + 2\xi A_0^2)(\alpha K^2 + 2\xi B_0^2) - 16\zeta^2 A_0^2 B_0^2] = 0, \quad (3)$$

i.e.

$$\Omega = \pm \left\{ \alpha K^2 [\alpha K^2 + \xi(A_0^2 + B_0^2)] \pm \right. \\ \left. \pm \alpha K^2 \sqrt{16\zeta^2 A_0^2 B_0^2 + \xi^2(A_0^2 - B_0^2)^2} \right\}^{1/2}. \quad (4)$$

We now investigate the modulational instability of incoherent crossing sea states by applying a Wigner transform to the wave amplitudes A and B . For a given function $f(t, x)$, the corresponding Wigner function $\rho_f(t, x, p)$ is defined as the Fourier transform of the two-point correlation function, i.e. [19–25]

$$\rho_f(t, x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu e^{i p \mu} \langle f^*(t, x + \mu/2) f(t, x - \mu/2) \rangle, \quad (5)$$

where the asterisk stands for the complex conjugate and the angular bracket denotes the ensemble average [26].

The function ρ_f is a generalized distribution function for the waves of the sea states. The wave intensity I_f corresponding to the function f can then be written as

$$I_f = \langle |f|^2 \rangle = \int dp \rho_f(t, x, p). \quad (6)$$

Applying the time derivative to Eq. (5), with $f = A$ or B , and using the nonlinear Schrödinger equations (1) and (2), valid for slowly varying envelopes, respectively, we obtain the nonlinearly coupled wave-kinetic equations

$$\frac{\partial \rho_A}{\partial t} + 2\alpha p \frac{\partial \rho_A}{\partial x} - 2(\xi I_A + 2\zeta I_B) \sin \left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p} \right) \rho_A = 0, \quad (7)$$

and

$$\frac{\partial \rho_B}{\partial t} + 2\alpha p \frac{\partial \rho_B}{\partial x} - 2(\xi I_B + 2\zeta I_A) \sin \left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p} \right) \rho_B = 0, \quad (8)$$

where the sin-operator is defined in terms of its Taylor expansion and the arrows give the direction of the differentiation. Equations (7) and (8) model the nonlinear evolution of partially coherent crossing sea states.

First order perturbations of (7) and (8) can be treated by letting $\rho_j(t, x, p) = \rho_{j0}(p) + \rho_{j1}(p) \exp(iKx - i\Omega t)$ and $I_j(t, x) = I_{j0} + I_{j1} \exp(iKx - i\Omega t)$, where $j = A, B$, $|\rho_{j1}| \ll \rho_{j0}$, and $|I_{j1}| \ll I_{j0}$. Linearizing with respect to the perturbation variables, we obtain the dispersion relation from Eqs. (7) and (8)

$$(1 + \xi \Delta_A)(1 + \xi \Delta_B) - 4\zeta^2 \Delta_A \Delta_B = 0, \quad (9)$$

where

$$\Delta_j = \int dp \frac{\rho_{j0}(p + K/2) - \rho_{j0}(p - K/2)}{\Omega - 2\alpha K p}. \quad (10)$$

Here we have used $2i \sin(iK/2 \partial_p) \rho_{j0}(p) = -\rho_{j0}(p + K/2) + \rho_{j0}(p - K/2)$. The dispersion relation (9) with (10) is valid for partially coherent crossing sea states that interact nonlinearly.

In the coherent case, the background waves are simple plane waves, so that the distribution functions are given by $\rho_{j0} = I_{j0} \delta(p)$, where $I_{A0} = A_0^2$ and $I_{B0} = B_0^2$. Inserting the coherent background distribution function into (10), the dispersion relation (9) reduces to Eq. (4).

We now assume that the sea states suffer from random perturbations, such that they have partial phase coherence. Thus means that the phase $\varphi_j(x)$ of the background wave amplitudes A, B satisfies $\langle \exp[-i\varphi_j(x +$

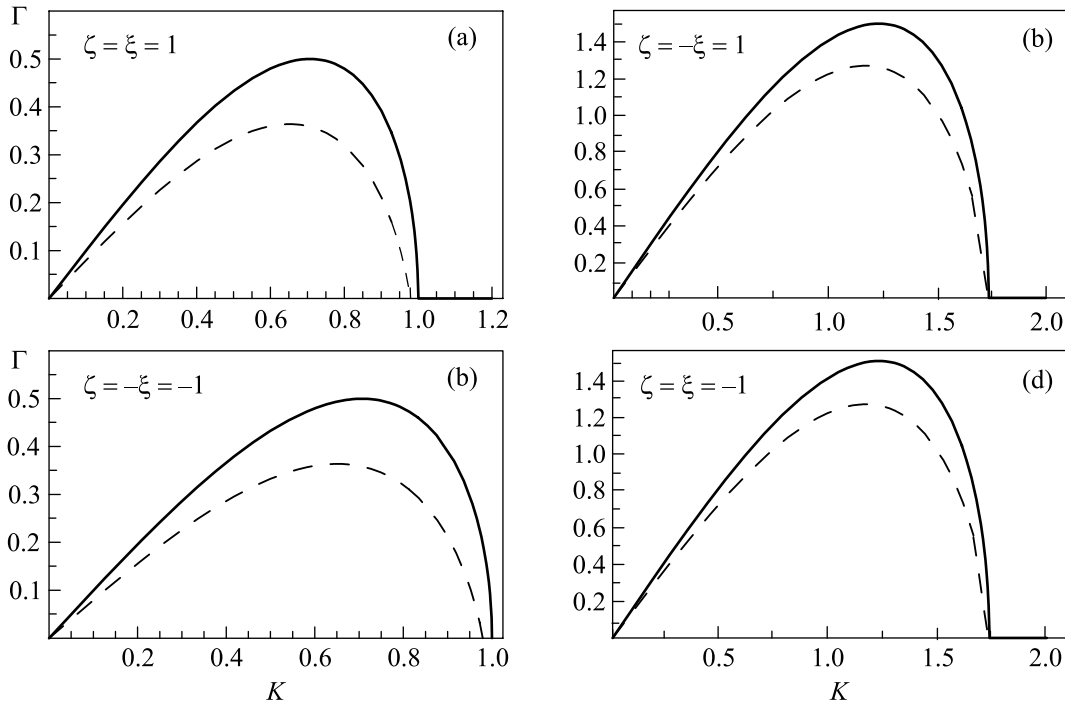


Fig.1. The effect of partial coherence when α is normalized to unity and $I_{A0} = I_{B0} = 0.5$. In all the panels, the coherent (full) and incoherent (dashed) cases are compared. The partial coherence stabilizes the wave modulation, i.e. the wave will be able to interact over longer distances before the perturbations grow to appreciable levels forming freak waves. In fact, for a wide enough spectral distribution, the modulational instability can be *completely suppressed*

$+ \mu/2) \exp[i\varphi_j(x - \mu/2)]\rangle = \exp(-p_{jW}|\mu|)$, corresponding to the Lorentz distribution function [27]

$$\rho_{j0}(p) = \frac{I_{j0}}{\pi} \frac{p_{jW}}{p^2 + p_{jW}^2}, \quad (11)$$

where p_{jW} represents the width of the j th distribution function. Thus, we see that the partial coherence in the waves phases give rises to a spectral broadening in terms of the distribution function. With this spectral background of sea states, the dispersion relation (9) takes the form

$$\begin{aligned} & \left[1 - \frac{2\xi\alpha K^2 I_{A0}}{(\Omega + 2i\alpha K p_{AW})^2 - \alpha^2 K^4} \right] \times \\ & \times \left[1 - \frac{2\xi\alpha K^2 I_{B0}}{(\Omega + 2i\alpha K p_{BW})^2 - \alpha^2 K^4} \right] - \\ & - 16\xi^2\alpha^2 K^4 I_{A0} I_{B0} / \{[(\Omega + 2i\alpha K p_{AW})^2 - \\ & - \alpha^2 K^4][(\Omega + 2i\alpha K p_{BW})^2 - \alpha^2 K^4]\} = 0. \quad (12) \end{aligned}$$

We next analyze the dispersion relation (12) by letting, for simplicity, $p_{AW} = p_{BW} = p_W$. We then obtain

$$\begin{aligned} \Omega = & -2i\alpha K p_W \pm \left\{ \alpha K^2 [\alpha K^2 + \xi(I_{A0} + I_{B0})] \pm \right. \\ & \left. \pm \alpha K^2 \sqrt{16\xi^2 I_{A0} I_{B0} + \xi^2 (I_{A0} - I_{B0})^2} \right\}^{1/2}, \quad (13) \end{aligned}$$

which agrees with (4) when $p_W \rightarrow 0$. We may then transform to dimensionless variables by $\Omega \rightarrow \Omega/\sqrt{\alpha|\xi|}$, $\alpha K^2 \rightarrow \alpha K^2/\sqrt{\alpha|\xi|}$, $\alpha K p_W \rightarrow \alpha K p_W/\sqrt{\alpha|\xi|}$, $I_{j0} \rightarrow I_{j0}\sqrt{|\xi|/\alpha}$, and $\zeta \rightarrow \zeta/\xi$. With these re-scalings, the group velocity dispersion $\alpha = 1$, while the self-nonlinearity coefficient becomes $\xi \rightarrow \pm 1$. In Figs.1 and 2 we have displayed the dimensionless growth rate $\Gamma = -i\Omega$ using Eq. (13). In Fig.1 we have chosen $I_{A0} = I_{B0} = 0.5$, while in Fig.2 we have $I_{A0} = 5I_{B0} = 0.5$. The lowering of the growth rate due to the spectral broadening of the background sea states ρ_{j0} can clearly be seen. Thus, the effect of the partial coherence of the background sea states is to stabilize their dynamics, allowing for long distance propagation of envelope solitons without the growth of freak waves. It should be noted that in general the spectral broadening enables interaction between the ocean waves over longer distances before any initial perturbation grows to an appreciable size. However, for large enough spectral width p_W the modulational instability growth rate may be completely suppressed. The results for $p_W = 0$, i.e. for coherent background waves without spatial spectral broadening, are consistent with the results presented by Onorato, Osborne, and Serio [17] (see the full curves in Figs.1 and 2), where it was found that crossing sea states have

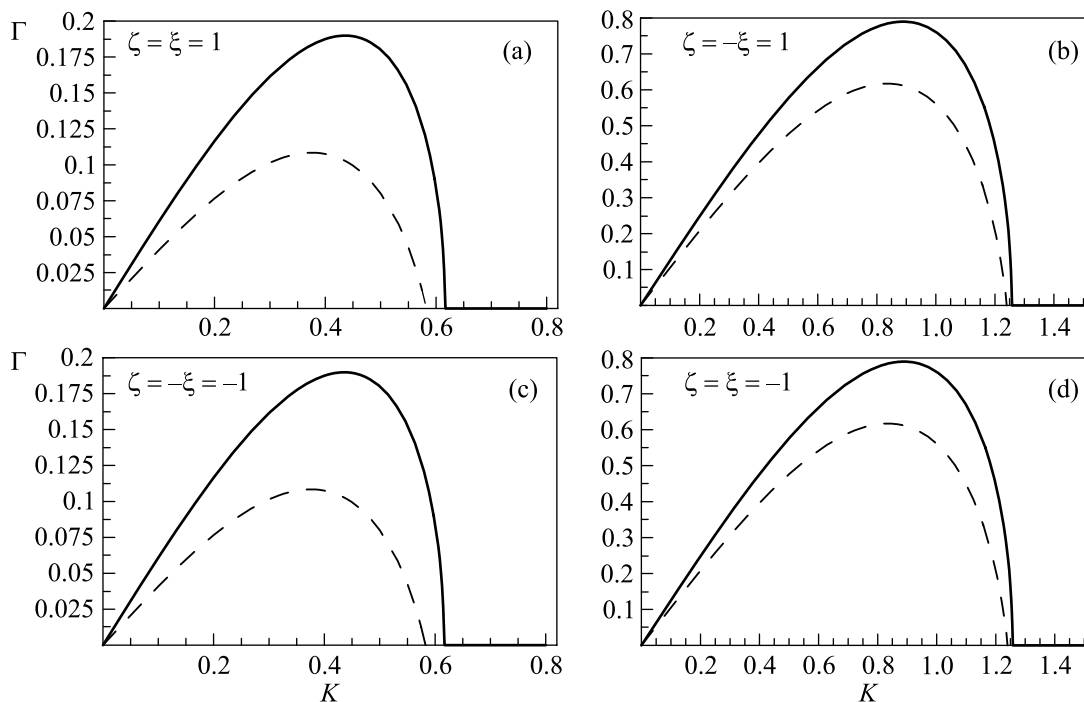


Fig.2. The effect of partial coherence when α is normalized to unity and $I_{A0} = 5I_{B0} = 0.5$. In all the panels, the coherent (full) and incoherent (dashed) cases are compared. As in the cases in Fig. 1, the partial coherence stabilizes the sea state modulation

a larger growth rate for the instability as compared to the single sea state instability.

Indeed, there are soliton solutions to the system (1) and (2). Following Ref. [28], we can find the dark-bright soliton pair

$$A(t, x) = A_0 \tanh[X(t, x)] \exp(i\kappa x - i\delta t) \quad (14)$$

and

$$B(t, x) = B_0 \operatorname{sech}[X(t, x)] \exp(i\kappa x - i\delta t), \quad (15)$$

where $X = (x - Vt)/L$, $V = 2\alpha\kappa$, $A_0^2 = B_0^2$, and $\delta = \alpha\kappa^2 + \xi A_0^2$. Here A_0 and L are treated as two free parameters determining the elevation and width of the solitons, respectively. Of course, the classical nonlinear Schrödinger equation has both soliton and multi-soliton solutions. Here we see that also the co-existence of dark and bright soliton water waves is possible.

To summarize, we have presented an investigation of the modulational instability of two incoherent crossing sea states that are nonlinearly interacting in deep water. For this purpose, we have introduced the Wigner transformation of the coupled nonlinear Schrödinger equations of Ref. [17] and obtained two coupled wave-kinetic equations (or von Neumann equations). The latter have been analyzed to obtain a nonlinear dispersion relation for background sea states that have broadband spectra, i.e. finite spectral width. It is found that

the growth rate of the modulational instability is suppressed. Hence, random phased nonlinearly interacting waves could propagate over long distances without being much affected by the modulational instability, and for a wide enough spectral distribution the formation of freak waves is thus completely suppressed. However, it should be stressed that this complete suppression is a result of the above NLS model calculations. As an alternative, one could start from the original equations, and then consider the interaction of wave packets which are eigenfunctions of the corresponding linearised problem with a background of Stokes wave trains. In that case, a random mixture of the eigenfunctions will postpone, although not completely suppress, the freak wave appearance.

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1. R. Smith, *J. Fluid Mech.* **77**, 417 (1976).
2. R. G. Dean, *Water Wave Kinetics*, Eds. A. Torum and O. T. Gudmestad, Kluwer, Dordrecht, 1990, p. 609.
3. C. Kharif and E. Pelinovsky, *Eur. J. Mech. B* **22**, 603 (2003).
4. B. V. Divinsky, B. V. Levin, and L. I. Lopatikhin et al., *Dokl. Earth Sci.* **395**, 438 (2004).

5. K. B. Dysthe, Proc. Roy. Soc. A **369**, 105 (1979); K. Trulsen and K. B. Dysthe, Wave Motion **24**, 281 (1996).
6. M. Onorato, A. R. Osborne, M. Serio, and S. Bertone, Phys. Rev. Lett. **86**, 5831 (2001).
7. A. A. Kurkin and E. N. Pelinovsky, *Facts, Theory and Modelling*, Nizhnii Novgorod University, 2004.
8. A. I. Dyachenko and V. E. Zakharov, JETP Lett. **81**, 255 (2005).
9. J. L. Hammack, D. M. Henderson, and H. Segur, J. Fluid Mech. **532**, 1 (2005).
10. V. E. Zakharov, J. Appl. Mech. Tech. Phys. **9**, 190 (1968).
11. F. Fedele, J. Offshore Mech. Arctic Eng. **128**, 11 (2006).
12. T. B. Benjamin and J. E. Feir, J. Fluid Mech. **27**, 417 (1967).
13. E. R. Tracy and H. H. Chen, Phys. Rev. A **37**, 815 (1988).
14. V. E. Zakharov, F. Dias, and A. N. Pushkarev, Phys. Rep. **398**, 1 (2004).
15. D. R. Crawford, P. G. Saffman, and H. C. Yuen, Wave Motion **2**, 1 (1980).
16. K. B. Dysthe, K. Trulsen, H. E. Krogstad, and H. Socquet-Juglard, J. Fluid Mech. **478**, 1 (2003).
17. M. Onorato, A. R. Osborne, and M. Serio, Phys. Rev. Lett. **96**, 014503 (2006).
18. H. C. Yuen and B. M. Lake, Adv. Appl. Mech. **22**, 67 (1982).
19. E. P. Wigner, Phys. Rev. **40**, 749 (1932).
20. J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).
21. I. E. Alber, Proc. R. Soc. A **363**, 525 (1978).
22. R. Fedele and D. Anderson, J. Opt. B: Quantum Semi-class. Opt **2**, 207 (2000).
23. R. Fedele, D. Anderson, and M. Lisak, Phys. Scr. T **84**, 27 (2000).
24. J. T. Mendonça, *Theory of Photon Acceleration*, Institute of Physics Publishing, Bristol, 2001.
25. M. Onorato, A. R. Osborne, R. Fedele, and M. Serio, Phys. Rev. E **67**, 046305 (2003).
26. Yu. L. Klimontovich, *The Statistical Theory of Non-Equilibrium Processes in a Plasma*, Pergamon Press, Oxford, 1967.
27. R. Loudon, *The Quantum Theory of Light*, Oxford Univ. Press, Oxford, 2000.
28. S. V. Vladimirov, L. Stenflo, and M. Y. Yu, Phys. Lett. A **153**, 144 (1991).