Josephson effect in thin films: the role of vortex excitations

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We consider quantum slips of phase at a round hole punctured in a thin superconducting film and show that virtual vortex pairs provide an efficient pathway for these processes. Specifically, in the limit when the normal-state resistivity of the film is large, the presence of the film causes at most a logarithmic interaction between phase slips. This is in contrast to the nearly linear confining interaction (and the consequent nearly activated behavior of the resistance) obtained when vortices are neglected.

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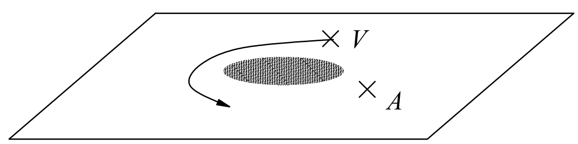
1. Introduction. Any superconducting device at a non-zero temperature is to some extent resistive. (In this article, we define resistance as dV/dI at zero current.) Of considerable theoretical and practical interest is the law according to which the resistance vanishes as the temperature T goes to zero. There are several cases in which the resistance has been theoretically predicted to obey a power law, $dV/dI \propto T^{\alpha}$; in all these cases, the exponent α is determined by dissipative effects that induce a logarithmic interaction between quantum phase slips.

One case is a resistively shunted Josephson junction (RSJ), where the interaction between the phase slips is controlled by the shunt resistance [1]. Another is a junction connecting two one-dimensional ($d = 1$) superconducting wires. In that case, α is determined by the phase stiffness of the leads [2] and is related to the line impedance of the wires – a dissipative effect due to the presence of the gapless plasmon mode [3, 4]. Similar physics has been predicted to occur in a junction coupled to a transmission line [5] and in a $d = 1$ wire without a junction but in the presence of disorder [6].

It is natural to ask what happens in another case when the plasmon in the leads is gapless [3], and there is no shunt, namely, when a junction connects two thin-film superconductors ($d = 2$). In recent papers, Hermele *et al.* [7] have reached the unexpected conclusion that in this case the resistance drops at $T \to 0$ much sharper than a power law, exhibiting a nearly activated behavior. Taken at face value, their result means that the presence of the film creates an almost unsurmountable obstacle for a change of the phase at the junction's end.

In this article, we show that the result of Ref. [7] is a consequence of the Gaussian approximation used by these authors to describe fluctuations in the leads. We will see that there is an efficient non-Gaussian mechanism for changing the phase. It involves nucleation, motion, and subsequent annihilation of virtual vortex pairs. In the limit when the normal-state resistivity of the film is large enough (so that the Ohmic dissipation at the vortex cores can be neglected), such pairs cause at most a logarithmic interaction between phase slips and, consequently, at most a power-law resistance. Put differently, our result means that, at sufficiently low temperatures, vortices in the leads *must* be involved if a phase slip at the junction is to occur with any appreciable probability.

2. The boundary problem. As a model of a thin-film lead, we consider a punctured superconducting plane – a plane with a round hole of radius R centered at the origin (Figure). A Josephson junction (not shown in the figure) will have the form of a tube attached to the puncture and connecting the system either to another film or to a bulk superconductor.



A punctured superconducting plane, with a vortex (V) and an antivortex (A). The vortex making a full circle around the puncture changes the phase at it by 2π

Away from vortex cores, the film can be described by a phase-only theory. We consider in detail the case when the charge-charge interactions are short-ranged (i.e., a ground plane is present nearby); the case of unscreened Coulomb interaction can be treated similarly. The Euclidean Lagrangian of the phase-only theory is

$$\mathcal{L}_E = in_0\partial_\tau\theta + \frac{1}{2g}(\partial_\tau\theta)^2 + \frac{K_s}{2}|\nabla\theta|^2 , \qquad (1)$$

where n_0 is the equilibrium superconducting density, g is the charge-charge coupling, K_s is the superconduct-

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ing stiffness, and $\tau = it$ is the Euclidean time. If we want to take vortices into account, it is essential to allow the phase $\theta(x, y; \tau)$ to be a multi-valued function of (x, y). For a single-valued θ, Eq. (1) becomes a wave Lagrangian describing a gapless plasmon with speed $c_0 = \sqrt{gK_s}$.

After a QPS in the junction occurs, the phase in the leads needs to relax to the profile corresponding to the new value of the supercurrent (the current is eventually replenished by an external battery, resulting in a nonzero voltage across the sample). Our goal will be to see how the matrix element responsible for this relaxation affects the phase-slip probability. For this calculation, we will use the semiclassical technique based on considering configurations that connect the relevant states in the imaginary (Euclidean) time – *instantons*. These instantons live in the leads: we will not need to specify the internal dynamics of the junction (due to the junction capacitance, etc.), beyond assuming that it does not suppress QPS too strongly.

An instanton will need to produce a phase difference between the puncture and the spatial infinity. The phase change at infinity during the relaxation process (i.e., after the phase slip, in Euclidean time) is zero, so the phase must change at the puncture. Accordingly, the boundary condition is taken in the form

$$\theta(r = R, \phi; \tau) = \theta_0(\tau), \qquad (2)$$

where $\theta_0(\tau)$ is a function of the Euclidean time but not of the polar angle ϕ. This means that there is a single phase that characterizes this side of the junction, i.e., the junction is point-like rather than extended. In the instanton considered in Ref. [7], $\theta_0(\tau)$ acts as an antenna that emits plasmons into the film, and what these authors show, in effect, is that the impedance matching of this antenna to the film is quite poor. In the instanton proposed here, plasmons are emitted by a vortex moving around the puncture.

3. Instantons from plasmons. To set the stage, let us first reproduce (by a different method) the result of Ref. [7], which is obtained by neglecting vortices in the film. In this case, the first (topological) term in Eq. (1) can be dropped, and the field θ can be assumed single-valued. The theory then becomes Gaussian, and the solution to the boundary problem is obtained as a linear combination of individual harmonics, plus a time-independent term that corresponds to a constant supercurrent I flowing out of the puncture. At $T = 0$,

$$\theta(r, \tau) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} e^{-i\Omega\tau} \tilde{\theta}_0(\Omega) \frac{K_0(kr)}{K_0(kR)} + \frac{I}{4\pi e K_s} \ln \frac{r}{R}, \qquad (3)$$

where $\tilde{\theta}_0(\Omega)$ is the Fourier transform of $\theta_0(\tau)$:

$$\tilde{\theta}_0(\Omega) = \int_{-\infty}^{\infty} d\tau e^{i\Omega\tau} \theta_0(\tau) ,$$

$k = |\Omega|/c_0$, where c_0 is the plasmon speed; K_0 is the modified Bessel function. Towards the end of the calculation, we will see how a finite temperature can be taken into account.

In the last term in Eq. (3), $e < 0$ is the electron charge. For definiteness, we take $I < 0$, i.e., electrons flow out, so that the ratio in which I and e will always occur is positive, $I/e > 0$.

We begin with the case when the stiffness K_s in one of the leads is much larger than in the other, so only fluctuations in the weaker lead need to be considered.

Substituting Eq. (3) into the Euclidean action, we find that, when the characteristic k are small, $kR \ll 1$, the main contribution comes from the gradient term in (1), and the action (relative to that for $\theta_0 = 0$) is

$$S_E \approx \frac{K_s}{2} \int_{-\infty}^{\infty} d\Omega |\tilde{\theta}_0(\Omega)|^2 \frac{1}{|\ln(kR)|} - \frac{I}{2e} \int_{-\infty}^{\infty} d\tau \theta_0(\tau) . \qquad (4)$$

Instead of a single instanton, changing θ_0 by 2π, it is somewhat more convenient to consider an instanton-antiinstanton (IA) pair

$$\theta_0(\tau) = 2\pi\Theta(\tau)\Theta(\tau_0 - \tau) , \qquad (5)$$

which rotates the phase by 2π at $\tau = 0$ and by -2π at $\tau = \tau_0 > 0$. We will not need to resolve the short-time details of these phase rotations, so the sharp step-functions will be sufficient. For this configuration, the action (4) becomes

$$S_E = 4\pi^2 K_s \int_{-\infty}^{\infty} d\Omega \frac{1 - \cos(\Omega\tau_0)}{\Omega^2 |\ln(kR)|} - \frac{\pi}{e} I\tau_0. \qquad (6)$$

The second term reflects the fact that an instanton releases an amount of energy, $E = \pi I/e$, from the supercurrent. This term is *deconfining*, i.e., it favors large values of the IA separation τ_0. On the other hand, the first term is *confining*, and it scales nearly linearly with τ_0 at large τ_0.

The power dissipated in the system is given by the energy E times the difference between the direct (instanton) rate $\mathcal{R}_+(I)$, and the reverse (antiinstanton) rate $\mathcal{R}_-(I)$. The voltage, obtained by dividing the power by the current, is

$$V = \frac{\pi}{e}(\mathcal{R}_+ - \mathcal{R}_-). \qquad (7)$$

At $T \neq 0$, instead of a single IA pair we must consider a chain of such pairs, periodic in τ with period $\beta = 1/T$.

This results in replacing the integral over Ω in Eq. (6) by a sum over $\Omega_n = 2\pi n T$, where n is a nonzero integer.

In the leading semiclassical approximation, the instanton rate is

$$\mathcal{R}_+(I) \sim \mathrm{Im} \int_0^\beta d\tau_0 e^{-S_E - S'} \qquad (8)$$

(a version of the optical theorem). Here S_E is the action (6), with the integral replaced by the sum, while S' is the part of the IA action unrelated to the presence of the film. In accordance with the way the problem was formulated in Sect. 2, we assume that, at large τ_0, S' is much smaller than the first term in Eq. (6).

The two terms in Eq. (6) become comparable at

$$\tau_0 \sim \frac{R}{c_0} e^{4\pi^2 e K_s / I} \equiv \tau_s,$$

which is very large at small currents. In the limit $\beta \ll \tau_s$, the saddle point that determines the imaginary part in (8) is located at $\tau_0 \approx \beta/2$, and, computing the sum over n to logarithmic accuracy, we obtain

$$\mathcal{R}_\pm(I) \sim \exp\left(-\frac{\pi^3 K_s}{T \ln(c_0 \beta / R)} \pm \frac{\pi I}{2eT} \right). \qquad (9)$$

When the leads are of comparable stiffnesses, K_s and K'_s, the phase change of 2π is shared between them, and K_s in Eq. (9) needs to be replaced with $K_s K'_s/(K_s + K'_s)$.

The resistance can now be obtained from Eq. (7), and we see that it is proportional to the rate (9) taken at $I = 0$. This has a nearly activated dependence on T ("nearly" means that the activation exponent is suppressed by a logarithm) – the result of Ref. [7]. We also see that this result can be interpreted as a consequence of the nearly linear confinement of instantons in the Gaussian theory.

4. Instantons from vortices. We now take vortices into account. Consider first a single vortex at distance r_0 from the origin. If

$$r_0 - R \gg \xi, \qquad (10)$$

where ξ is the vortex core radius, we can continue to use the phase-only theory (1), provided we allow θ to be multi-valued. Of course, vortices encircling the puncture at these relatively large r_0 do not necessarily make the best instantons; in all likelihood, smaller circles are more advantageous. What we intend to show, however, is that at sufficiently low T even these larger circles have much smaller Euclidean actions than the "vortexless" instantons considered in the preceding section.

It is useful to consider, in addition to θ, its dual field ψ, defined by

$$\frac{\partial \psi}{\partial x} = \frac{\partial \theta}{\partial y}, \qquad \frac{\partial \psi}{\partial y} = -\frac{\partial \theta}{\partial x}.$$

Since these are the Cauchy-Riemann conditions, the complex field

$$w = \psi + i\theta$$

is an analytic function of $z = x + iy$.

For a single vortex at the point z_0 of the complete (not punctured) plane, w equals $w_0 = \ln(z - z_0)$. The presence of the puncture will deform this into

$$w_1 + w_\Gamma = \ln(z - z_0) + \ln\left(1 - \frac{R^2}{z z_0^*}\right) + i\frac{\Gamma}{2\pi} \ln\frac{z}{R}, \qquad (11)$$

where w_1 is comprised by the first two terms, and w_Γ is the last term, describing a supercurrent I flowing out of the puncture;

$$\Gamma = \frac{I}{2eK_s}.$$

Note that Eq. (11) satisfies the equation of motion of the phase-only theory and obeys the boundary condition

$$\left.\frac{\partial \theta}{\partial \phi}\right|_{r=R} = 0. \qquad (12)$$

This is precisely the condition that the entire puncture has the same value of the phase, $\theta(r = R, \phi) = \theta_0$, and one readily sees that, by supplying the vortex position with a Euclidean time dependence, one can obtain any history $\theta_0(\tau)$ that may be required by Eq. (2).

In terms of the dual field ψ, the vortex is a "charge", i.e., a point at which $\nabla^2 \psi \neq 0$. The boundary condition (12) is equivalent to

$$\left.\frac{\partial \psi}{\partial r}\right|_{r=R} = 0.$$

Thus, ψ can be interpreted as the velocity potential of an ideal incompressible fluid incident from a source at $z = z_0$ and flowing past a round obstacle of radius R. In this picture, Γ is the circulation of the fluid around the obstacle.

We now construct a family of trial instanton configurations, each of which represents a three-step process: nucleation of a vortex-antivortex pair, motion of the vortex in a loop around the puncture (while the antivortex remains at rest), and annihilation of the pair. The solution corresponding to an antivortex at point z'_0 is

$$w_2 = -\ln(z - z'_0) - \ln\left(1 - \frac{R^2}{z z'^*_0}\right),$$

and the solution corresponding to a pair is $w = w_1 + w_2 + w_\Gamma$. For the phase θ, this gives

$$\theta(x, y; \tau) = \theta_1(x, y; \tau) + \theta_2(x, y) + \theta_\Gamma(x, y), \qquad (13)$$

where θ_1, θ_2, and θ_Γ are the imaginary parts of w_1, w_2, and w_Γ, respectively. In Eq. (13), we have made θ_1 τ-dependent by supplying a τ-dependence to the vortex position z_0.

Each instance of the three-step activity is a *single* instanton. An IA *pair* consists of an instanton (i.e., a vortex loop) at time around $\tau = 0$ and an antiinstanton (an antivortex making a loop) at time around $\tau = \tau_0 > 0$.

Since the vortex loop releases energy from the supercurrent, we expect a deconfining, linear in τ_0 term in the IA action – a counterpart of the second term in Eq. (6). However, we will see that none of the other terms in the action is nearly as sensitive to τ_0. In other words, there is no counterpart to the first, confining term of Eq. (6).

The deconfining term comes from the product of $\nabla(\theta_1 + \theta_2)$ and $\nabla\theta_\Gamma$, obtained when Eq. (13) is substituted in the Lagrangian (1). The calculation requires some care, because the resulting Lagrange function is not a single-valued function of the vortex position. However, the gradient of the Lagrange function (the force) must be. Accordingly, we first compute the force directly:

$$F_x = -K_s \int dx dy \nabla\theta_\Gamma \frac{\partial}{\partial x_0} \nabla\theta_1 = -K_s \Gamma \frac{y_0}{x_0^2 + y_0^2}, \qquad (14)$$

$$F_y = K_s \Gamma \frac{x_0}{x_0^2 + y_0^2}, \qquad (15)$$

and then integrate with respect to x_0, y_0 to obtain the action

$$S_{KZ} = -K_s \Gamma \int d\tau \arg[z_0(\tau)]. \qquad (16)$$

In the equivalent incompressible fluid, described by ψ, the force (14), (15) is the Kutta-Zhukovskii (KZ) lift force or, more precisely, the reaction force acting back on the source.

When R is of order of or smaller than ξ, the case of main interest to us, Eq. (10) implies $r_0 \gg R$. The remaining terms in the single-instanton action will be given in this limit, as indicated by the approximate equality sings in the equations below. The weakness of their dependence on τ_0—the crucial property that we seek to establish – does not depend on this approximation. The individual terms are as follows.

(i) The topological term in (1) gives rise to the Magnus force; the corresponding term in the action is

$$S_M = i n_0 \int d^2 x d\tau \partial_\tau \theta_1 \approx 2\pi i n_0 \int d\tau x_0 \dot{y}_0. \qquad (17)$$

Hereinafter overdots denote derivatives with respect to the Euclidean time τ.

(ii) The remaining terms in (1) are responsible, in addition to the lift-force action (16), for the kinetic energy and the vortex-antivortex potential: to logarithmic accuracy,

$$S_1 \approx \frac{\pi}{2g} \int d\tau (\dot{x}_0^2 + \dot{y}_0^2) \ln \frac{L}{\xi} + 2\pi K_s \int d\tau \ln \frac{l(\tau)}{\xi},$$

where $l(\tau)$ is the distance between the vortex and the antivortex, and L is the infrared cutoff – the smaller of the size of the film and the perpendicular magnetic penetration length $\lambda_\perp$. The appearance of the infrared cutoff has to do with the fact that in our trial configuration (13) the vortex moves as a rigid structure. We do not exclude that, when L far exceeds the plasmon wavelength $c_0\tau_0$, there are better trial configurations, which include retardation and for which L is replaced by $c_0\tau_0$. That would lead to a logarithmic dependence of the action on the IA separation, similar to that in the one-dimensional case.

It is well known that both the Magnus force and the inertial mass can be strongly renormalized by electrons at the vortex core [8]. The present calculation refers to the hydrodynamic limit, when the electron relaxation time τ_r is much shorter than either the timescale of the vortex motion or the inverse of the "minigap" of the core electrons.[2)] For short τ_r, the correction to the inertial mass is small, but the correction to the Magnus force is anomalously large and almost precisely cancels [8] the hydrodynamic contribution (17).

The deconfining term (16) is the only term in the instanton action that significantly depends on the presence of an antiinstanton at $\tau = \tau_0$. The Magnus force action is finite at $\tau_0 \to \infty$, and the kinetic term grows at most logarithmically (if the replacement of L with $c_0\tau_0$ contemplated above is effected). Accordingly, at small enough currents, the counterpart of Eq. (9) for the instanton rate is

$$\mathcal{R}_\pm(I) \sim \exp\left\{-S'(T) - S''(T) \pm \frac{\pi I}{2eT}\right\}, \qquad (18)$$

where $S'(T)$ is, as before, unrelated to the presence of the leads, while $S''(T)$ grows at most logarithmically with $1/T$. Thus, the resistance, obtained using Eq. (7), does not suffer from the nearly exponential in $1/T$ suppression that was characteristic of the "vortexless" instantons considered in the preceding section.

2) In this sense, it is analogous to the calculation [6] of the QPS rate in a $d = 1$ wire in the limit when the main dissipative effect is the finite wave impedance of the wire, rather than the normal conductance of the phase-slip core.

5. Discussion. We have considered, for the case of a punctured superconducting film, tunneling paths formed by virtual vortex-antivortex pairs encircling the puncture and found that the action S'' in the tunneling rate (Eq. (18)) depends at most logarithmically on T in the limit $T \to 0$. This means that, when a Josephson junction connects two such films (or a thin film and a bulk superconductor), the thin-film leads do not create too much of an obstacle to destruction of superconductivity by quantum phase slips. In particular, they do not cause the resistance to drop exponentially (or nearly exponentially) in $1/T$ at low temperatures.

One should keep in mind, though, that our results were obtained in the extreme hydrodynamic limit $\tau_r \to \to 0$, when the normal resistivity ρ of the film is effectively infinite. A finite ρ will cause dissipation at the vortex cores, and we cannot exclude that in this case a new long-range interaction between QPS will appear. We have shown, however, that at least the dramatic suppression of QPS obtained in the Gaussian theory [7] disappears when one takes vortices into account.

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