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Entanglement in a Two-Boson Coupled System

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A model for two nondegenerate cavity fields coupled through a reservoir is considered. Such a model can be employed for the study of Raman scattering in which the Stokes and anti-Stokes fields indirectly interact via a bath of phonon. An analysis for appearance of quantum entanglement between the fields and evaluation of the entanglement measure depending on initial states of the fields and on a state of the phonon reservoir is given.

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1. Introduction. In quantum information entangled states play a key role providing a quantum channel for communication. The nature of these states consists in quantum correlations that are generated in coupled systems. One of such systems is the Raman scattering which still remains a powerful method for spectroscopy in condensed matter physics. Quantum-mechanical description of the Raman effect holds the production of scattered radiation by the interaction of an incident laser beam with vibrational modes of a medium giving rise to correlations between the fields under interaction. Statistical and nonclassical properties of Raman scattering have been considered in a number of papers, including calculations of quantum factorial moments of the Stokes and anti-Stokes fields for estimation of photocount statistics [1, 2] as well as investigation of higher-order squeezing [3].

The study of quantum correlations of composite systems poses a problem of the quantitative assessment of entanglement. The measure of entanglement between the Stokes and a single phonon modes in the deterministic approach to the Raman effect has been estimated in [4]. Quantum treatment of Raman scattering with the stochastic coupling of the Stokes and anti-Stokes fields through a phonon bath has been given in [5] that made possible to take account of the damping of optical phonons. In [6] it has been proposed a computable measure of entanglement suited for a composite system in a mixed state as an usual result of interaction with a reservoir.

This paper aims to study the conditions of quantum entanglement initiation and the entanglement measure [6] between the Stokes and anti-Stokes fields in the Raman scattering based on the model [5] with regard to the bath state.

2. Model of Raman scattering. The model Hamiltonian for the production of the Stokes and anti-Stokes modes with frequencies ω_S and ω_A respectively by an incident laser field e_L can be represented as the following

$$\hat{H} = \hbar\omega_S \hat{a}_S^\dagger \hat{a}_S + \hbar\omega_A \hat{a}_A^\dagger \hat{a}_A + \sum_l \hbar\omega_{Bl} \hat{a}_{Bl}^\dagger \hat{a}_{Bl} - \\ - \sum_l (\hbar g_l e_L \hat{a}_S^\dagger \hat{a}_{Bl}^\dagger + \hbar\kappa_l^* e_L \hat{a}_A^\dagger \hat{a}_{Bl} + \text{h.c.}), \qquad (1)$$

where $\hat{a}_S$, $\hat{a}_A$ and $\hat{a}_{Bl}$ are the corresponding annihilation operators for the Stokes, anti-Stokes and phonon modes. Here the laser field is treated as undepleted by the Raman interaction and described by the form $e_L = E_L \exp(-i\omega_L t)$ with the constant laser amplitude E_L.

The solution of the Heisenberg-Langevin equations for the slowly varying operators $\hat{A}_j(t) = \hat{a}_j(t) e^{i\omega_j t}$ $(j = S, A)$ at the frequency resonance $2\omega_L = \omega_S + \omega_A$ is given as [2]

$$\hat{A}_S(t) = u_S(t)\hat{a}_S + v_S(t)\hat{a}_A^\dagger + \sum_l w_{Sl}(t)\hat{a}_{Bl}^\dagger, \\ \hat{A}_A(t) = u_A(t)\hat{a}_A + v_A(t)\hat{a}_S^\dagger + \sum_l w_{Al}(t)\hat{a}_{Bl}, \qquad (2)$$

where the notations $\hat{a}_j \equiv \hat{a}_j(0)$ are used. Time-dependent functions obey the following equalities

$$|u_S(t)|^2 - |v_S(t)|^2 - \sum_l |w_{Sl}(t)|^2 = 1, \\ |u_A(t)|^2 - |v_A(t)|^2 + \sum_l |w_{Al}(t)|^2 = 1, \qquad (3) \\ u_S(t)v_A(t) - v_S(t)u_A(t) - \sum_l w_{Sl}(t)w_{Al}(t) = 0$$

that come from the equal-time commutation relations of the boson operators $\hat{A}_j$. These equalities considered together with explicit forms of the functions

$$u_S(t) = \frac{1}{\gamma_S - \gamma_A}(\gamma_S e^{\Gamma t} - \gamma_A),$$
$$u_A(t) = \frac{1}{\gamma_S - \gamma_A}(\gamma_S - \gamma_A e^{\Gamma t}), \qquad (4)$$
$$v_S(t) = -v_A(t) = \frac{\sqrt{\gamma_S\gamma_A}}{\gamma_S - \gamma_A}(e^{\Gamma t} - 1)e^{i\phi},$$

determine the time-evolution of the model. In Eq. (4) the parameter $\Gamma = (\gamma_S - \gamma_A)|E_L|^2/2$ is defined by the damping constants $\gamma_S = 2\pi|g(\omega_B)|^2\rho(\omega_B)$ and $\gamma_A = 2\pi|\kappa(\omega_B)|^2\rho(\omega_B)$ for the reservoir spectrum $\rho(\omega_{Bl})$ at $\omega_B \simeq \omega_L - \omega_S$, and the phase $\phi = 2\phi_L + \phi_S - \phi_A$ is fixed by complex phases of the laser amplitude E_L, coupling constants for the Stokes g and anti-Stokes κ processes.

3. The Wigner function. In order to describe the dynamics of the Stokes and anti-Stokes fields, let us define their joint Wigner function. It can be done by introducing the symmetric characteristic function for these fields

$$\chi_{\text{sym}}(\beta_S, \beta_A, t) = = \text{Tr}\{\hat{\varrho}(0)\ \exp[\beta_S \hat{A}_S(t)^\dagger + \beta_A \hat{A}_A^\dagger(t) - \text{h.c.}]\}, \qquad (5)$$

where $\hat{\varrho}(0)$ is the initial density matrix of all fields in the Hamiltonian (1). From an experimental point of view, it is reasonable to assume that the Stokes and anti-Stokes fields can be initially found in superpositions of coherent and chaotic states that might correspond to the case of stimulating scattering, whereas the phonon bath is in a chaotic state. It means that the initial density matrix is represented in the factorized form and reads as

$$\hat{\varrho}(0) = \prod_{j=S,A} \frac{\langle n_j\rangle^{\hat{c}_j^\dagger \hat{c}_j}}{(1+\langle n_j\rangle)^{\hat{c}_j^\dagger \hat{c}_j + 1}} \prod_l \frac{\langle n_B\rangle^{\hat{a}_{Bl}^\dagger \hat{a}_{Bl}}}{(1+\langle n_B\rangle)^{\hat{a}_{Bl}^\dagger \hat{a}_{Bl}+1}}. \qquad (6)$$

In Eq. (6) there are introduced new operators $\hat{c}_{S,A} = \hat{a}_{S,A} - \xi_{S,A}$, where ξ_S and ξ_A are initial coherent amplitudes of the Stokes and anti-Stokes fields, and the averages $\langle n_j\rangle$ $(j = S, A, B)$ identify with mean numbers of bosons in the corresponding chaotic states. Then the Wigner function can be obtained in the form of Gaussian distribution

$$W(\alpha_S, \alpha_A, t) = = \frac{1}{\pi^4}\int d^2\beta_S d^2\beta_A \exp[(\alpha_S\beta_S^* - \alpha_S^*\beta_S) + (\alpha_A\beta_A^* - \alpha_A^*\beta_A)] \times \chi_{\text{sym}}(\beta_S, \beta_A, t) = = \frac{1}{\pi^2\mathcal{N}(t)}\exp\{-\frac{1}{\mathcal{N}(t)}\,(M_A(t)|\alpha_S - \xi_S(t)|^2 + M_S(t)|\alpha_A - \xi_A(t)|^2 - [N_{SA}^*(t)(\alpha_S - \xi_S(t))(\alpha_A - \xi_A(t)) + \text{c.c.}])\}, \qquad (7)$$

where

$$\xi_S(t) = u_S(t)\xi_S + v_S(t)\xi_A^*,$$
$$\xi_A(t) = u_A(t)\xi_A + v_A(t)\xi_S^*, \qquad (8)$$

and

$$M_S(t) = B_S(t) + \frac{1}{2} + \langle n_S\rangle|u_S(t)|^2 + \langle n_A\rangle|v_S(t)|^2,$$
$$M_A(t) = B_A(t) + \frac{1}{2} + \langle n_A\rangle|u_A(t)|^2 + \langle n_S\rangle|v_A(t)|^2, \qquad (9)$$
$$N_{SA}(t) = D_{SA}(t) + \langle n_S\rangle u_S(t)v_A(t) + \langle n_A\rangle u_A(t)v_S(t),$$

with

$$B_S(t) = (\langle n_B\rangle + 1)(|u_S(t)|^2 - 1) - \langle n_B\rangle|v_S(t)|^2,$$
$$B_A(t) = (\langle n_B\rangle + 1)|v_A(t)|^2 - \langle n_B\rangle(|u_A(t)|^2 - 1), \qquad (10)$$
$$D_{SA}(t) = (\langle n_B\rangle + 1)u_S(t)v_A(t) - \langle n_B\rangle v_S(t)u_A(t),$$

and the normalization

$$\mathcal{N}(t) = M_S(t)M_A(t) - |N_{SA}(t)|^2. \qquad (11)$$

4. Entanglement of the Stokes and anti-Stokes states. For investigation of entanglement initiation in the Stokes–anti-Stokes subsystem, we will apply the measure of entanglement that was suggested in [6]. This measure is based on negative eigenvalues of the partial transpose of the subsystem density matrix. In case of the Gaussian density operator, the negativity is completely defined by the symplectic spectrum of the partial transpose of the covariance matrix and thus can be presented in the analytical form. The Wigner function (7) is characterized by the following covariance matrix

$$\mathbf{V} \equiv \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{Z}^T & \mathbf{Y} \end{pmatrix} = = \left(\begin{array}{cc|cc} M_S & 0 & \text{Re}N_{SA} & \text{Im}N_{SA} \\ 0 & M_S & \text{Im}N_{SA} & -\text{Re}N_{SA} \\ \hline \text{Re}N_{SA} & \text{Im}N_{SA} & M_A & 0 \\ \text{Im}N_{SA} & -\text{Re}N_{SA} & 0 & M_A \end{array}\right). \qquad (12)$$

Then the logarithmic negativity

$$E = -\frac{1}{2}\log_2[4f(V)], \qquad (13)$$

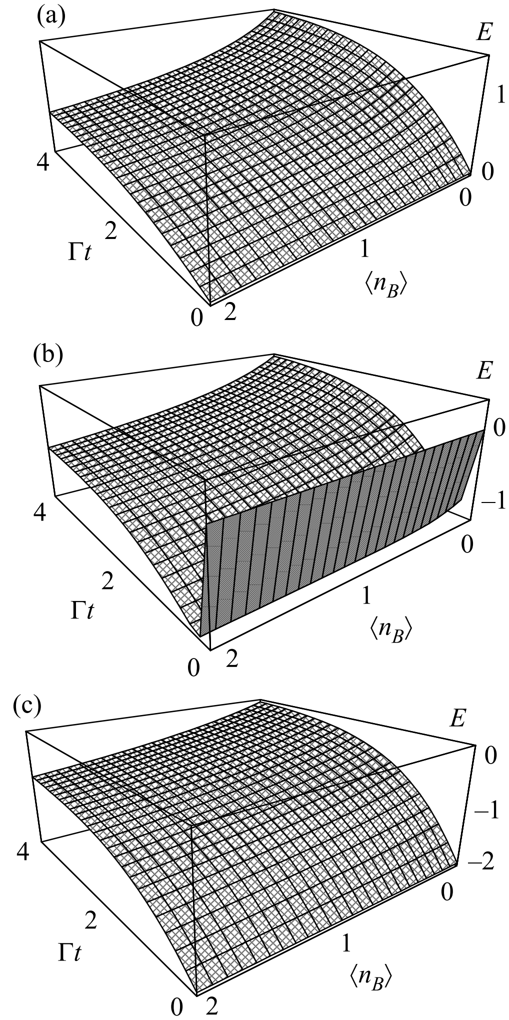
where

$$f(V) = (\det \mathbf{X} + \det \mathbf{Y})/2 - \det \mathbf{Z} - \\ - \sqrt{[(\det \mathbf{X} + \det \mathbf{Y})/2 - \det \mathbf{Z}]^2 - \det \mathbf{V}} = \\ = \frac{1}{2}\left[M_S^2 + M_A^2 + 2|N_{SA}|^2 - \right. \\ \left. - (M_S + M_A)\sqrt{(M_S - M_A)^2 + 4|N_{SA}|^2}\right], \quad (14)$$

will determine the strength of entanglement for $E > 0$. If $E \leq 0$, then the state is appeared to be separable. It follows from Eqs. (13), (14) and (9), (10) that the logarithmic negativity does not depend on the initial coherent amplitudes $\xi_{S,A}$ being charged with the displacement of the Wigner function (7) but only on the mean numbers of bosons $\langle n_j \rangle$ $(j = S, A, B)$ in the chaotic states which determine the width of the Gaussian distribution.

The behavior of the logarithmic negativity E for various initial states of the Stokes and anti-Stokes fields is represented in Figure. If the Stokes and anti-Stokes fields were in pure coherent states (Figure a), the state of their subsystem becomes entangled ($E > 0$) from the very outset of the interaction and the strength of entanglement monotonously increases in time approaching its asymptotic value. The noise of the phonon reservoir is appeared to work for destruction of entanglement tending to reduce its strength. When one of the fields (Stokes) was in a coherent state whereas a state of another one (anti-Stokes) had a chaotic constituent (Figure b) or states of both the fields contained chaotic admixtures (Figure c), the entanglement in the subsystem can arise only after certain interaction time. Moreover, if the phonon noise is large enough, then the entanglement might not happen between the fields at all (e.g., when $\langle n_B \rangle > 0.5$, in Figure b,c E holds below zero). Therefore, the phonon bath here comes to be a source of both quantum correlations and their decoherence.

It is interesting to note that correlations arising in this model were shown to lead not to a quadrature squeezing of the Stokes and anti-Stokes fields but to their higher-order squeezing of the sum-type [3]. A quantitative analysis of the time for the beginning of a sum squeezing in comparison with that required for entanglement initiation here shows a good agreement. Thus, the entanglement in this model can be a manifestation of higher-order quantum correlations like a sum squeezing rather than associates with an ordinary quadrature squeezing.



Logarithmic negativity E as a function of the scaled interaction time Γt and the bath noise $\langle n_B \rangle$ at $\gamma_S = 2\gamma_A = 2$ for: (a) both the fields are in pure coherent states; (b) the Stokes field is in a coherent state and the anti-Stokes field is in a state with a chaotic admixture of $\langle n_A \rangle = 2$; (c) both the fields are in states having chaotic constituents with $\langle n_S \rangle = \langle n_A \rangle = 2$

5. Conclusion. In this paper, a model for the interaction between two boson fields coupled by a reservoir has been considered. The model was treated for description of Raman scattering in which the Stokes and anti-Stokes fields were produced. The phenomenon of entanglement between the Stokes and anti-Stokes fields was found due to their indirect interaction through the phonon bath. The beginning of entanglement for initial chaotic states of the fields, in contrast to the coherent ones, requires a fixed interaction time. At the same time, a noise of the phonon bath diminishes the strength of entanglement and can even completely destroy entanglement. It is worth, however, noting that the model assumption of undepleted laser beam (the parametric approximation) is valid only for short interaction times or weak interactions, so that the given inferences on the properties of entanglement cannot be directly extended to the general case beyond the limits of this approach.

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