Scattering by a Dirac monopole in a global bundle space

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The wave function for the scattering of a charged particle by a Dirac monopole has been found in a global bundle space.

The Dirac monopole, which is of interest in many physics problems, can be described, as we know, in terms of bundles[1,2] with a global space $P = \mathbb{C}^2 \setminus 0$ and fiber bundle U(1) (Refs. 3–5). The group U(1) acts on P in accordance with $(z^1,z^2) \to (z^1 e^{i\alpha}, z^2 e^{i\alpha})$, which corresponds to the charge $n = 2eg = 1$. The monopoles with other $n \in \mathbb{Z}$ can be described from the given one by factorizing P with respect to the discrete group Z_n (Refs. 1, 6, and 7)

In our problem the bundle P with $n = 1$ is universal. The projection onto the base $|R^3 \setminus 0$ $p{:}\mathbb{C}^2 \setminus 0 \to |R^3 \setminus 0$ is given by the equation $p(z,\bar{z}) = \bar{z}\sigma_i z = x_i$. The form of the

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connection on such a bundle is given by the expression[4]

$$\omega = (\bar{z}\,dz - z\,d\bar{z})/2\bar{z}z. \tag{1}$$

Knowing the action of the group U(1), it is easy to find the vector v tangent to the bundle:

$$\frac{d}{d\alpha}\,\Psi(ze^{i\alpha},\,\bar{z}\,e^{-i\alpha})\,|_{\alpha=0} = iv\Psi(z,\bar{z}), \qquad v = z\partial - \bar{z}\,\bar{\partial}. \tag{2}$$

The value $\omega(iv) = i$—it is equal to the generator of the U(1) Lie group. From (1) it is possible to find the horizontal vectors h_i, $i = 1,2,3$, i.e., such vectors that $\omega(h_i) = 0$, which upon the projection of p are projected on $\partial_i = \partial/\partial x_i$:

$$h_i = \frac{1}{2\bar{z}z}\,(\bar{z}\,\sigma_i\,\bar{\partial} + \partial\,\sigma_i z), \qquad ph_i = \partial_i. \tag{3}$$

The operators h_i and v satisfy the relations

$$[h_i, v] = 0, \qquad [h_i, h_j] = i\Omega_{ij} v,$$

where $\Omega_{ij} = \frac{1}{2}\epsilon_{ijk}\bar{z}\sigma_k z/(\bar{z}z)^3$ is the monopole field strength, and $\Omega = d\omega = \frac{1}{2}\Omega_{ij}dx^i \Lambda dx^j$.

Solov'ev[4] considered the Lagrangian mechanics of a point in a bundle space, which gave rise, after quantization, to a Schrödinger equation with a Hamiltonian specified in the global bundle space. The alternative method which we are proposing is characterized by the fact that the same result can be obtained immediately from the global variational principle by considering the functional: the quantum-mechanical "action" determined in the bundle space P. After some obvious changes, this principle can be extended to field-theory problems.

We apply

$$S = \int \Psi^*(i\partial_t + \frac{1}{2\mu}\,h_i h_i)\,\Psi\,dt\,\Lambda\,dV, \tag{4}$$

where Ψ is the wave function on P and $dV = (\bar{z}z/\pi)dz^1\Lambda dz^2\Lambda d\bar{z}^1\Lambda d\bar{z}^2$ is a volume element on P which is invariant under the group U(1) and which determines the scalar product (Φ,Ψ) of the wave functions. Any Ψ can be expanded in the U(1) form

$$\Psi = \sum_{n=-\infty}^{+\infty} \Psi_n,$$

$$\Psi_n = P_n\Psi = \int_0^{2\pi} e^{-in\alpha}\,\Psi(ze^{i\alpha},\,\bar{z}\,e^{-i\alpha})\,\frac{d\alpha}{2\pi}$$

$$P_n^2 = P_n, \qquad P_n P_m = 0, \qquad n \neq m. \tag{5}$$

It immediately follows from the invariance of dV that $\int\Psi_n^*\Psi_m dV = 0$ for $n \neq m$. It is easy to show (by choosing local coordinates of the direct product of the base and the bundle and then integrating over the bundle) that S corresponds to the standard formulation for the description of a charged particle in the monopole field on the base $\mathbb{R}^3\setminus 0$ with a charge n if $\Psi = \Psi_n$.

Varying (4) with respect to Ψ^*, we obtain the Schrödinger equation in the global bundle space p:

$$i\partial_t\Psi = -\frac{1}{2\mu}h_i h_i\Psi , \tag{6}$$

where $h_i h_i = (1/\bar{z}z)\partial\bar{\partial} + [1/4(\bar{z}z)^2]v^2$. It is clear that $[h_i,P_n] = [v,P_n] = vP_0 = 0$ and $v\Psi_n = n\Psi_n$. In addition, the operators P_n, v, and $-ih_i$ are Hermitian operators for sufficiently rapidly decreasing functions.

Let us consider a problem involving the determination of the wave functions describing the scattering of a charged particle in the global bundle space which strikes the monopole with a momentum **k**.[8]

We introduce for the momenta **k** the same bundle as P, i.e., $(q,\bar{q})\in\mathbb{C}^2\setminus 0$, $k_i = \bar{q}\sigma_i q$, which we denote as P^*. Taking into account the gauge invariance and the invariance under the rotation of the charge-monopole SU(2) system,[4,5] we easily see that the wave function determined on $P\times P^*$ should depend only on the variables of the type $\bar{q}z$, $\bar{z}q$, and $\bar{q}q\bar{z}z$.[1)] For $E = k^2/2\mu = (\bar{q}q)^2/2\mu$, the equation $(E + h_i h_i/2\mu)\Psi = 0$, reduces to the equation

$$\left(\partial_\rho^2 + \frac{2}{\rho}\partial_\rho + \frac{1}{\rho^2}\left[(1-\bar{\epsilon}\epsilon)\,\partial_\epsilon\partial_{\bar{\epsilon}} - \frac{1}{2}(\epsilon\partial_\epsilon + \bar{\epsilon}\partial_{\bar{\epsilon}})\right] + 1\right)\Psi = 0, \tag{7}$$

where $\rho = \bar{q}q\bar{z}z$, $\epsilon = \bar{q}z/\sqrt{\rho}$, and $\bar{\epsilon} = q\bar{z}/\sqrt{\rho}$. For $n \geqslant 0$ its solution is

$$\Psi_{hq\bar{q}} = \sum_{l=0}^{\infty} e^{i\alpha(l,n)}\,\epsilon^{|n|}\left(l + \frac{|n|+1}{2}\right)\sqrt{\frac{2\pi}{\rho}}\,J_{\nu(l,n)}(\rho)\,P_l^{(0,|n|)}(2\bar{\epsilon}\epsilon - 1); \tag{8}$$

for $n<0$, $\bar{\epsilon}^{|n|}$ replaces $\epsilon^{|n|}$; $\nu(l,n) = \{[l + (|n|+1)/2]^2 - n^2/4\}^{1/2}$.[2)] The numbers $\alpha(l,n) = (\pi/2)(2l - \nu(l,n) + \frac{1}{2})$ are determined under the condition that there is no converging wave in (8). The asymptotic behavior of (8), with $r = \bar{z}z \to \infty$, is given by the equation

$$\Psi_{nq\bar{q}} \approx \epsilon^{|n|}\left[e^{i\mathbf{kx}} + \frac{1}{r}\left(\sum_{l=0}^{\infty}\frac{\left(l + \frac{|n|+1}{2}\right)}{ik}\left(e^{i\pi(l-\nu(l,n)+1/2)} - 1\right)\right.\right.$$
$$\left.\left.\times\, P_l^{(0,|n|)}(2\bar{\epsilon}\epsilon - 1)\right)e^{ikr}\right], \tag{9}$$

which determines the scattering amplitude

$$f_n(\theta) = \sum_{l=0}^{\infty}\frac{l + \frac{|n|+1}{2}}{ik}\cos^{|n|}\frac{\theta}{2}\left(e^{i\pi(l-\nu(l,n)+1/2)} - 1\right)P_l^{(0,|n|)}(\cos\theta).$$

It is easy to verify that

$$\int \Psi^*_{nq\bar{q}} \Psi_{mq'\bar{q}'} dV = (2\pi)^3 \delta_{nm} \delta(\mathbf{k} - \mathbf{k}'), \tag{10}$$

$$k_i = \bar{q}\sigma_i q, \qquad k'_i = \bar{q}'\sigma_i q'.$$

The problem of the scattering on a monopole thus can effectively be solved by doubling the bundle $P \times P^*$ and using reduction (7).

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[1)]The groups SU(2) and U(1) act on $P \times P^*$ diagonally; specifically $g:P \times P^* \to gP \times gP^*$, where $g \in U(1)$ or SU(2).
[2)]$P_l^{(0,n)}(\beta) = (-i)^l/2^l l!(1+\beta)^{-n}(d^l/d\beta^l)[(1-\beta)^l(1+\beta)^{l+n}]$ is a Jacobi polynomial, where $P_l^{(0,0)} = P_l$; $J_{\nu(l,n)}$ is a Bessel function.

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