Dynamical properties of a finite one-dimensional conductor

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The frequency dependence of the conductivity of a one-dimensional disordered system of finite size z has been determined. The static dielectric constant of an open sample is found to be proportional to z^2 and much higher than that in an infinite system. The polarization of the sample initially increases as $\ln^3 t$ and then relaxes to a classical value as $\exp[-(1/z)\ln^2 t]$ as a function of time t. A relaxation of this type apparently is characteristic of both a metal and an insulator of arbitrary dimensions.

We know that the residual static conductivity of a one-dimensional disordered conductor decreases exponentially as its length z increases and that the conductivity $\sigma(\omega)$ at a low but finite frequency is primarily imaginary and linear in ω as $z \to \infty$. This behavior, for the occurrence of which only the electron states need be localized, is implied in the exact solution of the problem in the one-dimensional case. We will discuss here the conductivity when both ω and z are finite, ignoring the interaction between electrons.

The limit passages $\omega \to 0$ and $z \to \infty$ are nontrivial. The dielectric constant $\epsilon(\omega) = 4\pi \, \mathrm{Im}\, \sigma(\omega)/\omega$ of the insulator with $z = \infty$ and $\omega \to 0$, for example, is finite and proportional to the square of the localization length. With $\sigma(0) \neq 0$, when the electron can go from one localized state to another, the sample becomes completely polarized, however, in sufficient time. This means that $\epsilon(0)$ is the same as in a metal with the same state density, i.e., $\epsilon(0) \propto z^2$. The frequency dependence of the conductivity should describe the transition from a small "insulating-state" value for short times t to a "metallic-state" value in the limit $t \to \infty$.

To determine $\sigma(\omega)$ in a finite one-dimensional sample with open boundaries, we will use Berezinskiĭ's technique, in which

$$\frac{\sigma(0)}{\sigma_0} = \frac{2}{z l_2} \int_0^z dx \int_x^z dy \sum_{m, m'=0}^{\infty} [L_{m+1}(x) - L_m(x)] Z_{m,m'}(x,y) \times [L_{m'+1}(y-z) - L_{m'}(y-z)]; \quad (1)$$

here σ_0 is the Drude conductivity (with allowance for the spin degeneracy $\sigma_0 = 2e^2 l_2 h^{-1}$), and l_2 is the mean free path relative to the backward scattering. The functions L_m and $Z_{m,m}$, which describe the left (right) part and the central part of the conductivity diagrams, satisfy the equations[1]

$$\left(ms + \frac{d}{dx}\right) L_m = m^2 (L_{m+1} + L_{m-1} - 2L_m), \quad (2)$$

$$\left[\left(m + \frac{1}{2}\right) s - \frac{d}{dx}\right] Z_m = (m+1)^2 (Z_{m+1} - Z_m) - m^2 (Z_m - Z_{m-1}), \quad (3)$$

where $s = -2i\omega l_2 v_F^{-1}$, and v_F is the Fermi velocity; all the lengths are given in units of l_2. The boundary conditions for (2) and (3) at the open boundaries are[2,3]

$$L_m(x=0) = \delta_{m,0}; \quad Z_{m,m'}(x=y) = \delta_{mm'}. \quad (4)$$

For large values of m and m' essential for low frequencies ($s \ll 1$), the solution of Eq. (2) can be found by means of a Laplace transform

$$L_m(x) = m^{1/2} \int_{-\infty + i(1+0)}^{\infty + i(1+0)} \frac{d\lambda}{\pi} \varphi(\lambda) s^{-i\lambda/2} \exp\left(-x \frac{1+\lambda^2}{4}\right) K_{i\lambda}[2(ms)^{1/2}], \quad (5)$$

where $\varphi(\lambda) = (1/8)\Gamma^3(-i\lambda/2 - 1/2)\Gamma^{-2}(-i\lambda - 1)$, and $K_{i\lambda}(\xi)$ is the Bessel function. Solution (3) is found by means of Lebedev-Kantorovich transformation

$$Z_{m,m'}(x,y) = \int_{-\infty}^{\infty} \frac{\gamma \sinh \pi\gamma d\gamma}{2\pi^2 (mm')^{1/2}} K_{i\gamma}[2(ms)^{1/2}] K_{i\gamma}[2(m's)^{1/2}] \times \exp\left[-|x-y| \frac{1+\gamma^2}{4}\right]. \quad (6)$$

Substituting (5) and (6) into (1), we find

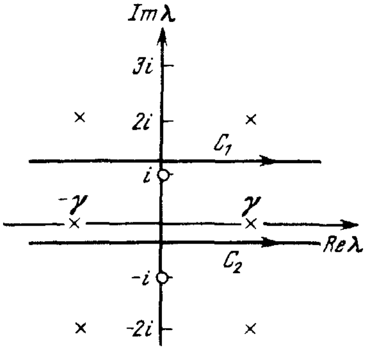

FIG. 1. Contours of integration C_1 and C_2 in Eq. (7). In the limits $\omega \to 0$ and $z = \text{const}$, C_1 and C_2 should be displaced by $+i\infty$, and the principal contribution comes from the poles shown by the crosses. In the other limits $\omega = \text{const}$ and $t \to \infty$, C_1 and C_2 should be dropped to the real axis, and the principal contribution to $\sigma(\omega)$ comes from the pole $\lambda = i$.

$$\frac{\sigma(\omega)}{\sigma_0} = \Big[\int_{C_1}(1+i\lambda_1^{-1}) + \int_{C_2}(1+i\lambda_1^{-1})\Big]d\lambda_1 \Big[\int_{C_1}(1+i\lambda_2^{-1}) + \int_{C_2}(1-i\lambda_2^{-1})\Big]d\lambda_2\, s^{-i(\lambda_1+\lambda_2)/2}$$
$$\times \frac{\varphi(\lambda_1)\varphi(\lambda_2)}{z(\lambda_1^2-\lambda_2^2)} \int_{-\infty}^{\infty} d\gamma \frac{[\psi(\lambda_1^2-\gamma^2)-\psi(\lambda_2^2-\gamma^2)]\sinh \pi\gamma}{(\cosh \pi\lambda_1 - \cosh \pi\gamma)(\cosh \pi\lambda_2 - \cosh \pi\gamma)} \exp\Big(-z\frac{1+\gamma^2}{4}\Big), \tag{7}$$

where $\psi(u) = [\exp(-zu/4) - 1]/u$, and the contours of integration $C_{1,2}$ are shown in Fig. 1. In the limit $z \to \infty$, the principal contribution to $\sigma(\omega)$ in the integration over $\lambda_{1,2}$ comes from the pole $\lambda = i$, giving the familiar result[1]

$$\sigma(\omega) = 4\pi s\sigma_0 \int_0^{\infty} (\gamma^2-1)(\gamma^2+1)^{-2}\left(\cosh\frac{\pi\gamma}{2}\right)^{-2} d\gamma$$
$$= -8i\zeta(3)e^2\omega l_2^2(v_F h)^{-1}. \tag{8}$$

If z is finite and $\omega \to 0$, $\sigma(\omega)$ is determined by the poles $\lambda = 2in \pm \gamma$; $n = 0,1,2\ldots$

$$\frac{\sigma(\omega)}{\sigma_0} = \left(\frac{\pi^5}{z}\right)^{1/2}\exp\left(-\frac{z}{4}\right) - \frac{\omega i}{6v_F} l_2 z^2$$
$$-\frac{z}{2}\sum_{n=2}^{\infty}\left(\frac{2i\omega l_2}{v_F}\right)^n \Gamma^6\left(\frac{n+1}{2}\right)\frac{\exp\left(z\frac{n^2-1}{4}\right)}{n(n-1)\Gamma^4(n)}. \tag{9}$$

The first term in (9) determines the static conductivity[2] and the second term determines the static dielectric constant $\epsilon(0) = (4\pi e^2/hv_F)(l_2 z)^2$ which does not depend on the disorder and which is determined strictly by the state density $2/hv_F$ and the sample size $l_2 z$. The third term on the right side of (9) describes the transition from (8) to this value. This transition can naturally be analyzed in a time-dependent representation.

Let us consider the time dependence of the dipole moment of the sample, $d(t)$, after the electric field E is turned on at $t=0$. According to (9), for $t\exp[-(z/2)]\ll\exp(z^{1/2})$ we have

$$\frac{d(t)}{d(\infty)} = 1-\delta(t);\quad \delta(t)=\frac{6}{(\pi z^3)^{1/2}}\left(1+\frac{1}{2\Lambda}\right)^2\frac{\Gamma^6(1+\Lambda)}{\Gamma^3(2+2\Lambda)}\exp(-z\Lambda^2), \tag{10}$$

$$d(\infty)=\sigma_0 E z^3\frac{l_2^2}{6v_F}=\frac{e^2E}{3v_F h}(zl_2)^3;\quad \Lambda=\frac{\ln t}{z}-\frac{1}{2}=\frac{1}{z}\ln(te^{-z/2}), \tag{11}$$

where t is measured in units of $2l_2v_F^{-1}$. In this case, therefore, in the limit $t\to\infty$, $d(t)$ relaxes to $d(\infty)$ in a "logarithmically normal" manner: faster than in the case of any power of t but slower than in the case of an exponential law. The main increase in $d(\infty)$ occurs at $t\ll\exp(z/2)$. If $t\ll\exp(z/2-z^{1/2})$, we have

$$d(t)=8z^{-3}d(\infty)(\ln^3 t+6\ln^2 t)-d(\infty)\delta(t). \tag{12}$$

We see that $d(t)$ increases as $\ln^3 t$ and does not depend on z. This behavior can be logically explained: In a time t only the interface regions of the sample, whose size is of order $\ln t$, have time to become polarized in accordance with (11). A comparison of (12) with the dipole moment corresponding to the Berezinskiĭ equation (8)

$$d_0(t)=\frac{4\zeta(3)zl_2^2}{v_F}\sigma_0 E= 24\zeta(3)z^{-2}d(\infty) \tag{13}$$

shows that all the results obtained for an infinite sample are valid only for $3\zeta(3)z\gg\ln^3 t$, i.e., for $t\ll\exp([3\zeta(3)z]^{1/3})$.

Equations (10) and (12) are valid for $\Lambda^2\gg(1/z)$. At the same time, if $\Lambda^2\ll1/4$, then $d(t)$ can be described by the expression

$$\frac{d(t)}{d(\infty)} = 1+[8z^{-3}(\ln^3 t+6\ln^2 t)-1]\,\Phi(-z^{1/2}\Lambda)-\frac{3}{(\pi z)^{1/2}}\exp(-z\Lambda^2) \tag{14}$$

$$\Phi(u)=\int_{-\infty}^{u}\exp(-w^2)dw/\pi^{1/2};\quad \Phi(0)=\frac{1}{2},$$

which is identical to (10) and (12) in the common regions of applicability.

The same method can be used to trace the spreading of the electron density in the finite sample. If at $t=0$ the electron is situated at the point y, then the probability that it will be found at the point x at the time t is

$$P(x,y;t)=\int_{-\infty}^{\infty}\frac{ds}{8\pi il_2}e^{st}\sum_{m,m'=0}^{\infty}[L_{m+1}(x)+L_m(x)]$$
$$\times\ Z_{m,m'}(x,y)\,[L_{m'+1}(z-y)+L_{m'}(z-y)]. \tag{15}$$

Substituting (5) and (6) into (15), we find, for $x=y=z/2$,

$$P(t) = \frac{1}{3l_2} \int_{C_1} \frac{d\lambda}{2\pi} \int_{C_2} \frac{d\gamma}{2\pi} \varphi(\lambda+\gamma)\varphi(\lambda-\gamma) |\Gamma(2+i\gamma)|^2 \Gamma(2-i\lambda)$$

$$\times \; t^{1+i\lambda} \exp\left(-z \frac{1+\lambda^2+\gamma^2}{4}\right). \qquad (16)$$

For $e^{-z/2} t \ll 1$ the principal contribution to (15) comes from the pole $\lambda = i \pm \gamma$. This leads to a known result[1,4,5] $P(t) = (3l_2)^{-1}[1 + 0(t^{-3})]$. For $e^{-z/2} t \gg 1$ integral (15) is determined by the saddle-point value $\lambda = i(2\Lambda + 1)$:

$$P(t) = \frac{1}{3l_2} \frac{1}{\pi z} \Gamma(3 + 2\Lambda)\varphi^2(i + 2i\Lambda)\exp(-z\Lambda^2). \qquad (17)$$

Persistent relaxation (10) and (17) means that the time it takes the electron to escape from the sample is a random value whose distribution obeys a normal logarithmic law. This value is typically $\eta^{-1} \sim \exp(\alpha x)$, where x is the distance to the nearest boundary, and α is the inverse localization length. If the distribution of the value α is assumed to be normal, then the distribution of the level width η and the time required for the particle to leave the sample η^{-1} will be logarithmically normal.

This distribution of the level widths makes it possible to explain the logarithmically normal distribution of the local state density.[3]

A logarithmically normal relaxation (10) and the law governing the increase in the $\exp(zn^2/4)$ coefficients of the expansion of the function $\sigma(\omega)$ in a series [Eq. (9)] are in very good agreement with the results of Ref. 6, in which the same quantities with an arbitrary dimensionality were studied in the metallic region by the effective field-theory method (the extended nonlinear σ model). Laws which differ from (9) and (10) in that z is replaced by a quantity proportional to $\ln[\sigma_0/\sigma(0)]$ in the arguments of the exponential functions were found to hold. The same agreement was previously found for the state density distribution function.[3,6] The persistent relaxation and the density-function tails of the mesoscopic fluctuations in the metal and insulator are very similar. We believe that this is as it should be: The atypical realizations of the random potential, which form these asymptotic expressions, sense the metal-insulator transition only slightly.

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