Dirac quantization conditions and the three-cocycle

A. B. Ryzhov
P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow

(Submitted 24 December 1987)
Pis'ma Zh. Eksp. Teor. Fiz. **47**, No. 3, 128–130 (10 February 1988)

With reference to the scattering of an electron by a Dirac monopole, it is shown that a quantization of the product of the charges of the electron and the monopole cannot be extracted from the condition that the corresponding 3-cocycle disappears.

In the problem of the scattering of an electron by a magnetic charge μ, the Dirac quantization condition $2e\mu = n$ is found by requiring that the splicing of the wave functions corresponding to different gauges of the vector potential of the monopole be single-valued. The splicing functions lead us to Wu-Yang stratifications,[1] which are labeled by integers n.

Jackiw,[2] Grossman,[3] and Wu and Zee[4] have constructed a "nonassociativity representation" of translations; a measure of the nonassociativity of this representation is a three-cocycle. The vanishing of the three-cocycle within 2π times an integer is precisely the Dirac quantization condition. In this letter, however, we show that (despite Ref. 2) the vanishing of the three-cocycle cannot be used to derive Dirac's condition, since the latter is implicitly contained in the original representation.

Let us recall how Dirac's quantization condition is obtained. Because of the unavoidable singularities in the vector potential of a monopole, there are singularities in the Schrödinger equation and in the wave function. We thus need at least a pair of vector-potential gauges, so that each point from $\mathbf{R}^3$-{O} will lie in a region in which at least one of the gauges is continuous. For example, with $A_r = A_\theta = 0$, $A_\varphi = (\mu/r)\tan(\theta/2)$, the lower $\theta = \pi$ semiaxis serves as a filament of singularities, while at $A_r = A_\theta = 0$, $A_\varphi = -(\mu/r)\cot(\theta/2)$, the upper semiaxis, $\theta = 0$, serves the same purpose. These two descriptions are distinguished on $2\mu\nabla\varphi(x)$, where φ is the azimuthal angle. In this case, however, the wave functions ψ and ψ' for Schrödinger equations corresponding to these forms of the vector potential are coupled by the gauge transformation $\psi(x) = \psi'(x)\exp(2ie\mu\varphi(x))$ in the region $\theta \neq 0, \pi$. The single-valuedness of the splicing function $\exp(2ie\mu\varphi)$ leads to Dirac quantization condition $2e\mu = n$.

The construction of the 3-cocycle is based on the following construction.[2-4] We assume that group G acts on the quantity x: $x \to gx$. We consider a representation of group G in the space of functions $\psi(x)$ of the form

$$U(g)\,\psi(x) = \mathscr{A}_1(x,g)\,\psi(gx).$$

We assume that the operators $\mathscr{A}_1$ do not obey the composition law in the group; i.e.,

$$U(g_1)U(g_2)\,\psi(x) = \mathscr{A}_2(x,g_1,g_2)\,U(g_2g_1)\,\psi(x)$$

or

$$\mathscr{A}_1(x,g_1)\,\mathscr{A}_1(g_1x,g_2) = \mathscr{A}_2(x,g_1,g_2)\,\mathscr{A}_1(x,g_2g_1). \tag{1}$$

We assume that the operators $\mathscr{A}_2$ commute and associate with the operators $\mathscr{A}_1$, but the operators $\mathscr{A}_1$ are not associative:

$$[\mathscr{A}_1(x,g_1)\,\mathscr{A}_1(g_1x,g_2)]\,\mathscr{A}_1(g_2g_1x,g_3) \tag{2}$$

$$= \exp(i\alpha_3(x,g_1,g_2,g_3))\,\mathscr{A}_1(x,g_1)[\mathscr{A}_1(g_1x,g_2)\,\mathscr{A}_1(g_2g_1x,g_3)].$$

In order to reconcile (2) with (1), we must require that α_3 satisfy certain conditions. These conditions can be found in the following way: We multiply (2) from the right by $\mathscr{A}_1(g_3g_2g_1x,g_4)$ and use (1) and (2) alternately. We find

$$\alpha_3(g_1x,g_2,g_3,g_4) - \alpha_3(x,g_2g_1,g_3,g_4) + \alpha_3(x,g_1,g_3g_2,g_4)$$

$$- \alpha_3(x,g_1,g_2,g_4g_3) + \alpha_3(x,g_1,g_2,g_3) = 0 \pmod{2\pi}.$$

This is the cocyclicity condition for α_3.

Let us recall the definition of a cocycle. We assume that the operator Δ sends the function $\alpha(x,g_1,\ldots,g_n)$ of n group arguments into the function $\Delta\alpha$, of $n+1$ group arguments, in accordance with the rule

$$\Delta\alpha(x,g_1,\ldots,g_n,g_{n+1}) = \alpha(g_1x,g_2,\ldots,g_{n+1}) - \alpha(x,g_2g_1,\ldots,g_{n+1})$$

$$+ \alpha(x,g_1,g_3g_2,g_4,\ldots,g_{n+1}) - \ldots + (-1)^n\,\alpha(x,g_1,\ldots,g_{n+1}g_n)$$

$$+ (-1)^{n+1}\,\alpha(x,g_1,\ldots,g_{n-1},g_n).$$

It is easy to verify the fundamental property $\Delta^2 = 0$. An n-cocycle is a function $\alpha(x,g_1,...,g_n)$ such that we have $\Delta\alpha = 0$. The cocycle α is trivial, or a coboundary, if it can be represented in the form $\alpha = \Delta\beta$.

We have found that under assumptions (1) and (2) regarding the operators $\mathscr{A}_1$ and $\mathscr{A}_2$ the function α_3 is a cocycle. The vanishing (within $2\pi n$) of cocycle α_3 means that this representation is associative.

We now consider this representation in connection with the problem of the scattering of a charged particle by a magnetic monopole. Instead of the ordinary action of the translation group,

$$U(\mathbf{a})\,\psi(\mathbf{r}) = \exp(i\mathbf{a}\mathbf{p})\,\psi(\mathbf{r}) = \psi(\mathbf{r}+\mathbf{a}),$$

we consider the action defined by means of the generalized-momentum operator $\mathbf{v} = \mathbf{p} - e\mathbf{A}(\mathbf{r})$, where $\mathbf{A}$ is the vector potential of the monopole:

$$U(\mathbf{a})\,\psi(\mathbf{r}) = \exp(i\mathbf{a}\mathbf{v})\,\psi(\mathbf{r}).$$

Writing

$$U(\mathbf{a})\,\psi(\mathbf{r}) = \exp(i\mathbf{a}\mathbf{v})\exp(-i\mathbf{a}\mathbf{p})\,\psi(\mathbf{r}+\mathbf{a}),$$

we find

$$\mathscr{A}_1(\mathbf{r},\mathbf{a}) = \exp\Big[-ie\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{a}} \mathbf{A}(s)ds\Big],$$

where the integration is carried out along the straight line connecting $\mathbf{r}$ and $\mathbf{r}+\mathbf{a}$. Furthermore, it follows from (1) that we have

$$\mathscr{A}_2(\mathbf{r},\mathbf{a}_1,\mathbf{a}_2) = \exp[-ie\Phi],$$

where Φ is the magnetic flux through the triangle $(\mathbf{r},\mathbf{r}+\mathbf{a}_1,\ \mathbf{r}+\mathbf{a}_1+\mathbf{a}_2)$. We now consider the tetrahedron $ABCD$ with the vertices $(\mathbf{r},\mathbf{r}+\mathbf{a}_1,\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2,\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)$, shown in Fig. 1. The left side of expression (2) can be rewritten as

$$\begin{aligned}
&[\mathscr{A}_1(\mathbf{r},\mathbf{a}_1)\,\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1,\ \mathbf{a}_2)]\,\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2,\mathbf{a}_3)\\
&= \exp(-ie\Phi(ABC))\,\mathscr{A}_1(\mathbf{r},\mathbf{a}_1+\mathbf{a}_2)\,\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2,\mathbf{a}_3)\\
&= \exp(-ie\Phi(ABC))\exp(-ie\Phi(ACD))\,\mathscr{A}_1(\mathbf{r},\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)
\end{aligned} \qquad (3)$$

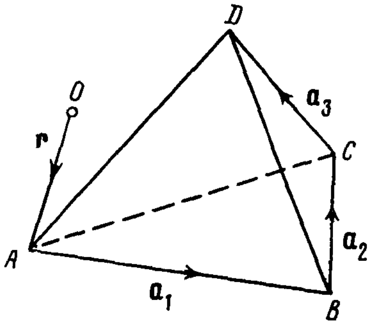

FIG. 1. The tetrahedron $ABCD$, defined by the radius vector $\mathbf{r}$ and by the displacements $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. The point 0 is the position of the monopole.

The right side of (2) can be transformed in a corresponding way:

$$\mathscr{A}_1(\mathbf{r},\mathbf{a}_1)[\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1,\mathbf{a}_2)\,\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2,\mathbf{a}_3)] \tag{4}$$

$$=\exp(i\alpha_3(\mathbf{r},\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3))\exp(-ie\Phi(BCD))\exp(-ie\Phi(ABD))\,\mathscr{A}_1(\mathbf{r},\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3).$$

Comparing (3) and (4), we find

$$\alpha_3 = -e\Phi(ABCD).$$

If tetrahedron $ABCD$ does not contain a monopole, either in its interior or on its boundary, then we have $\alpha_3 = -4\pi e\mu$. In the case $2e\mu = n$ we have $\alpha_3 = -2\pi n$, and the representation becomes associative. Conversely, the requirement of associativity leads to the condition that $e\mu$ is quantized.

We note that for a product of operators, understood as a composition of mappings, associativity always holds, by definition. What sort of operation would give rise to a 3-cocycle (a nonassociativity) here? In this scheme, the replacement of $\mathscr{A}_1(\mathbf{r},\mathbf{a}_1)\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1,\mathbf{a}_2)$ by $\mathscr{A}_2(\mathbf{r},\mathbf{a}_1,\mathbf{a}_2)\,\mathscr{A}_1(\mathbf{r},\mathbf{a}_1+\mathbf{a}_2)$ implies switching from a sum of integrals along the segments $[AB]$ and $[BC]$ to an integral over the segment $[AC]$, to which an integral of curl **A** over a triangle is added in accordance with Stokes's theorem. Jackiw,[2] Grossman,[3] and Wu and Zee[4] did not intend a global gauge (a single vector potential), since when the monopole is inside the tetrahedron, the filament of singularities will penetrate at least one face; for such a face, Stokes's theorem will not hold. In a local gauge this situation can be avoided only if singularities of two different gauges of the vector potential strike different pairs of faces. However, a local approach presupposes a splicing of the wave functions and thus Dirac's condition, as we have shown above.

Boulware *et al.*[5] called attention to the vanishing of the 3-cocycle. Mickelsson's comment[6] and Jackiw's response[6] deal with a mathematical refinement of this construction. The question of finding a new quantization method, in contrast, has not been taken up.

I wish to thank M. A. Solov'ev for calling my attention to Refs. 2–5, and I thank I. S. Shapiro and A. G. Savinkov for stimulating discussions.

[1]T. T. Wu and C. N. Yang, Phys. Rev. **D12**, 3845 (1975).
[2]R. Jackiw, Phys. Rev. Lett. **54**, 159 (1985).
[3]B. Grossman, Phys. Lett. **152B**, 93 (1985).
[4]Y.-S. Wu and A. Zee, Phys. Lett. **152B**, 98 (1985).
[5]D. G. Boulware, S. Deser, and B. Zumino, Phys. Lett. **153B**, 307 (1985).
[6]J. Mickelsson, Phys. Rev. Lett. **54**, 2379 (1985); R. Jackiw, Phys. Rev. Lett. **54**, 2380 (1985).

Translated by Dave Parsons