Two-loop contribution to the four-point function in superstring theory

A. Morozov

Institute of Theoretical and Experimental Physics, Academy of Sciences of the USSR

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Pis'ma Zh. Eksp.Teor. Fiz. **47**, No. 4, 181–184 (25 February 1988)

A prototype two-loop contribution to a four-point function, which satisfies the requirements of module invariance and finiteness, is constructed. A study of the factorization properties of the expressions proposed here may prove sufficient for an unambiguous determination of the correct result.

1. A significant effort has recently been made to construct a two-loop calculus in superstring theory.[1–3] It has been shown that under certain special conditions on the choice of odd modules the 0-, 1-, 2-, and 3-point functions vanish (at least for external boson states), in agreement with general expectations.[4] A more important step forward would be to calculate the first nonvanishing expression: the contribution to the 4-point function. In this letter we present some formulas which satisfy the conditions of module invariance and finiteness (and which are quite similar in structure to entities

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which arise in two-loop calculations) and which therefore qualify as prototypes of the actual expression for the four-point function. It is extremely likely that a study of the factorization properties of these formulas at the boundary of the module space (where they must coincide with a definite single-loop amplitude) will make it possible to unambiguously determine the correct answer.

2. Many of the calculations in the two-loop approximation in string theory have been carried out in hyperelliptic coordinates, so we, too, will use those coordinates (Ref. 5, for example). There is no difficulty in transforming to a representation in terms of, say, matrices of periods. In this representation, the role of module invariance is played by a projective symmetry. All natural entities are rational functions of z and $s(z) = [\Pi_{i=1}^{2p+2}(z-a_i)]^{1/2}$ (p is the type of surface). The projection transformations act in accordance with the following rules: $z \to (Az+B)/(Cz+D)$; $a_i \to (Aa_i+B)/(Ca_i+D)$; $s(z) \to s(z)/N^{1/2}(Cz+D)^3$, $N = \Pi_{i=1}^{2p+2}(Ca_i+D)$. Any difference $z-z'$ transforms uniformly: $z-z' \to (z-z')/(Cz+D)(Cz'+D)$.

The two-loop measure for a boson strong is $d\mu_{\rm bos} = 1/G^{13}|\Pi da/d\Omega/(\Pi(a))^3|^2$, where $\Pi_{i=1}^{2p+2}da_i/d\Omega$ is a projection-invariant measure [and $d\Omega = da'da''da'''/(a'-a'')(a''-a''')(a'''-a')$ is itself invariant]; $\Pi(a) = \Pi_{i<j}(a_i-a_j)$; and G replaces the ordinary $\det \mathrm{Im}\, T$ in this representation: $G = |\det\sigma|^2 \det \mathrm{Im}\, T = \int |(v_1-v_2)dv_1dv_2/s(v_1)s(v_2)|^2$. It is easy to verify that under a projection transformation we have $\Pi da/d\Omega \to N^{-2}\Pi da/d\Omega$; $\Pi(a) \to N^{-5}\Pi(a)$; $G \to |N|^2 G$, and $d\mu_{\rm bos}$ is invariant.

3. In a single loop we have $d\mu_{\rm bos} = 1/G_1^{14}|\Pi da/d\Omega/(\Pi(a))^3|^2$; only in this instance we have $G_1 = \int |dv/s(v)|^2$ and $G_1 \to |N| G_1$ under a projection transformation. This expression is also invariant [it is the same as $(\mathrm{Im}\,\tau)^{-14}|d\tau/\eta^{24}(\tau)|^2$].

The simplest nonvanishing entity in the case of superstring is the 4-point function $\langle \psi\tilde\psi(z_1)e^{ip_1X(z_1)} \ldots \psi\tilde\psi(z_4)e^{ip_4X(z_4)}\rangle$, and the corresponding measure is

$$\frac{1}{G_1^6}\left|\frac{\Pi da/d\Omega}{\Pi(a)}\ \frac{dz_1}{s(z_1)}\cdots\frac{dz_4}{s(z_4)}\right|^2 \langle e^{ip_1X(z_1)}\ldots e^{ip_4X(z_4)}\rangle \qquad (1)$$

[this is the same as the familar expression $(\mathrm{Im}\,\tau)^{-6}|d\tau\, d\xi_1 \ldots d\xi_4|^2 \times \langle e^{ip_1X(\xi_1)}\ldots e^{ip_4X(\xi_4)}\rangle$]. This expression is module-invariant. Furthermore, it leads to a *finite* result in an integration over module space. In principle, singularities might arise at the boundary of the module space, where we have $a_i - a_j \sim \sqrt{t_{ij}} \to 0$ for some $i \neq j$. In this limit we have $\Pi da/d\Omega/\Pi(a) \sim (dt/t) = d\ln t$, and $G_1 \sim \ln t$. The reason for the second asymptotic expression is that we have $s(v) \to (v-a_i)[\Pi_{k\neq i,j} \times (v-a_k)]^{1/2}$, so the integral determining G_1 diverges logarithmically. Corresponding logarithmic divergences arise from an integration over $dz_\alpha/s(z_\alpha)$. Accordingly (with $p_1 = \ldots = p_4 = 0$), the single-loop measure has the behavior $d\ln t/(\ln t)^6(\ln t)^4 \sim d\ln t/(\ln t)^2 = -d(1/\ln t)$, and $1/\ln t$ is a finite quantity in the limit $t \to 0$.

4. Making the corresponding transition from the boson partition function to a superstring 4-particle amplitude is more complicated. The direct analog of (1),

$$\frac{1}{G^5}\left|\frac{\Pi\, da/d\Omega}{\Pi(a)}\ \frac{dz_1}{s(z_1)}\ \cdots\ \frac{dz_4}{s(z_4)}\right|^2 \tag{2}$$

(here and below, $p_1 = ... = p_4 = 0$; it is not difficult to reconstruct the p dependence, but a certain accuracy would be required[6]), does not work because of projection invariance: This expression is multiplied by $|(Cz_1 + D)...(Cz + D)|^2$. It was suggested by Kallosh *et al.*[3] on the basis of a preliminary analysis of the Green-Schwarz formalism that (2) should be multiplied by

$$\left|\sum_{i\neq j}\frac{(z_1-a_i)(z_2-a_i)(z_3-a_j)(z_4-a_j)}{(a_i-a_j)^2} + \text{permutations of } z_1,z_2,z_3,z_4\right|^2 . \tag{3}$$

In particular, this step would restore the projection invariance. Unfortunately, multiplication by (3) gives rise to an additional pole of $1/t$ (the complete expression is $\sim dt/t^2$). This result contradicts finiteness.

Furthermore, an ansatz of this sort is not completely satisfactory from the standpoint of the *NSR* formalism, since it does not explicitly incorporate possible contributions from the $\langle \partial X\, \partial X\rangle$ part of the correlation function of the supercurrents. (Similar contributions may also arise in the Green-Schwarz approach.) To get an idea of the structure of these contributions, it is useful to take a look at the equation for the correlation function $\langle \partial X(a)\partial X(b)\rangle$ in hyperelliptic coordinates. In terms of the Grassmann fields ξ and η, which are 0- and 1-differentials, respectively, we find

$$\langle \partial X(a)\partial X(b)\rangle = \int \partial\xi(a)\partial\bar\xi(b) e^{\int\bar\xi\Delta\xi} D\bar\xi D\xi \Big/ \int e^{\int\bar\xi\Delta\xi} D\bar\xi D\xi + (a\leftrightarrow b)$$

$$= \int \partial\xi(a) e^{\int\eta\bar\partial\xi} D\xi D\eta \int \partial\bar\xi(b) e^{\int\bar\eta\partial\bar\xi} D\bar\xi D\bar\eta e^{\int\eta\bar\eta} \Big/ \int e^{\int\eta\bar\partial\xi} e^{\int\bar\eta\partial\bar\xi} e^{\int\eta\bar\eta} D\xi D\eta D\bar\xi D\bar\eta$$

$$+ (a\leftrightarrow b) = \int_{d^2v_1 d^2v_2} \langle \partial\xi(a)\eta(b)\eta(v_1)\eta(v_2)\rangle\langle\bar\eta(v_1)\bar\eta(v_2)\rangle$$

$$\Big/ \int_{d^2v_1 d^2v_2} \langle\eta(v_1)\eta(v_2)\rangle\langle\bar\eta(v_1)\bar\eta(v_2)\rangle$$

$$+ (a\leftrightarrow b) = \int\left\{\left|\frac{(v_1-v_2)dv_1dv_2}{s(v_1)s(v_2)}\right|^2 \frac{1}{s(b)}\frac{\partial}{\partial a}\left[\frac{(b-v_1)(b-v_2)}{(a-b)(a-v_1)(a-v_2)}\right.\right.$$

$$\times\left(s(a) + s(b)\frac{(a-v_1)(a-v_2)}{(b-v_1)(b-v_2)} + s(v_1)\frac{(a-b)(a-v_2)}{(v_1-b)(v_1-v_2)}\right.$$

$$\left.\left.\left. + s(v_2)\frac{(a-b)(a-v_1)}{(v_2-b)(v_2-v_1)}\right)\right]\right\} \Big/ \int\left|\frac{(v_1-v_2)dv_1dv_2}{s(v_1)s(v_2)}\right|^2 + (a\leftrightarrow b).$$

(the numerator in the last expression is the same as G). If the general arguments given in Ref. 2 are true, we conclude that the result contains only the residues of this correlation function at coincident points [i.e., some combination of the coefficients of $(a-b)^{-2}$, $(a-b)^{0}$, and $(a-b)^{2}$]. It would then be a simple matter to reach agreements with the argument that (2) should be multiplied not only by a structure of the type in (3) but also by (nonanalytic) expressions of the type

$$\left| \sum_{i\neq j} \frac{(z_1-a_i)(z_2-a_i)(z_3-a_j)(z_4-a_j)}{(a_i-a_j)^2} \int \left[\frac{(v_1-a_j)(v_2-a_i)}{(v_1-a_i)(v_2-a_j)} + (v_1 \longleftrightarrow v_2) \right] \right. \times \left| \frac{(v_1-v_2)dv_1dv_2}{s(v_1)s(v_2)} \right|^2 \Big/ G + \text{permutations of } z_1,z_2,z_3,z_4 \left. \right|^2 . \qquad (4)$$

Combining (3) and (4), we can cancel the extraneous $1/t$ pole and replace (2) by

$$\frac{1}{G^7} \left| \frac{\Pi da/d\Omega}{\Pi(a)} \frac{dz_1}{s(z_1)} \cdots \frac{dz_4}{s(z_4)} \int \sum_{i\neq j} \frac{(z_1-a_i)(z_2-a_i)(z_3-a_j)(z_4-a_j)}{(v_1-a_i)(v_1-a_j)(v_2-a_i)(v_2-a_j)} (v_1-v_2)^2 \right. \times \left| \frac{(v_1-v_2)dv_1dv_2}{s(v_1)\, s(v_2)} \right|^2 + \text{permutations of } z_1,z_2,z_3,z_4 \left. \right|^2 . \qquad (5)$$

We regard this expression as a prototype of the two-loop contribution to the four-point function. There are of course other expressions of this type with projection invariance [e.g., we could replace a_j in (5) by a_i and replace the double sum $\Sigma_{i\neq j}$ by the single sum Σ_i]. The factorization condition which relates the two-loop amplitude to the single-loop amplitude (with an additional dilaton insertion) should select the correct result.

5. Expressions of the type in (5), which look extremely natural from the standpoint of two-loop calculations, are not as good in the sense of finiteness as their single-loop analogs. These expressions do not have tachyon singularities, but the dilaton singularities are essential. The explanation is that (2) contains only the fifth power of G, so (5) has the behavior $[(d\ln t)/(\ln t)^5](\ln t)^4 = d(\ln\ln t)$. Fortunately, for nonvanishing external momenta, $p_\alpha p_\beta < 0$, contributions of boson correlation functions arise: $\exp p_\alpha p_\beta \langle X(z_\alpha)X(z_\beta)\rangle$. They vanish when z_α and z_β coincide. Consequently, at least one of the logarithmic divergences which appear in the limit $z_\alpha z_\beta \to a_i$ disappears, and the result turns out to be finite. These arguments appear to be very similar to those of Ref. 6.

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