Steady-state solution of the equation for the simple production and quadratic disappearance of gluons

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(Submitted 4 February 1988)
Pis'ma Zh. Eksp. Teor. Fiz. **47**, No. 6, 286–288 (25 March 1988)

Incorporating the possibility of a coalescence of gluons in a cascade results in a switch from a negative binomial multiplicity distribution to a Poisson distribution at high energies.

At high energies, hadron-hadron collisions involve so many partons that the latter cannot be treated as noninteracting. In addition to the cascade production of partons, it becomes necessary to consider the inverse process: the recombination of partons or the coalescence of several partons to form a single parton. According to Gribov *et al.*,[1] one can introduce an "overlap" parameter for partons (gluons, quarks) in quantum chromodynamics:

$$W(x, Q^2) = \alpha_s(Q^2)\, x F(x, Q^2) / (Q^2 R_h^2), \tag{1}$$

where $\alpha_S(Q^2)$ is the strong-interaction constant, Q^2 is the square of the momentum transfer, $F(x,Q^2)$ is the probability density of the number of partons with a fraction x of the hadron momentum at a given Q^2, and R_h is the radius of the hadron. The probability for the inverse recombination of partons is therefore proportional to the number of partons, and it becomes significant at $xF(x,Q^2)/(Q^2R_h^2) \approx 1$.

Let us examine the development of a purely gluon cascade with recombination. For simplicity, we restrict the analysis to the three-gluon vertices $g \to gg$ and $gg \to g$. We write a Kolmogorov-Chapman equation[2] for the probability $P_n(t)$, i.e., the probability for observing n gluons at a given value of the Evolution parameter of the system, t:

$$dP_n(t)/dt = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) - \mu_n P_n(t) + \mu_{n+1} P_{n+1}(t)\,, \qquad (2)$$

where λ_n (μ_n) is the probability that a system with n gluons will undergo a transition into a system with $n+1$ ($n-1$) gluons. Assuming the production of gluons to be independent in each $g\to gg$ fission event, we find the probability for the production of k new gluons from n existing gluons upon a change δt in the evolution parameter:

$$\lambda_n \delta t = C_n^k (\lambda\delta t)^k (1-\lambda\delta t)^{n-k}\,,$$

where λ is the constant at the $g\to gg$ vertex. This constant is independent of n. Under the condition $\lambda\delta t \ll 1$, the term with $k=1$ is obviously dominant, and we have $\lambda_n = n\lambda$—so-called simple production. Combinatorial considerations also show that the probability for the coalescence of two gluons into a single gluon is proportional to $C_n^2 = n(n-1)/2$, so we have $\mu_n = \mu n(n-1)$, where μ is the constant at the $gg\to g$ vertex. (In general, we would have $\mu \ll \lambda$ because of the difference in the phase spaces for the fission and coalescence processes.) We note that we have $\mu_n/\lambda_n \sim n$, according to QCD estimate (1). We can then rewrite Eq. (2) as

$$dP_n(t)/dt = -\lambda n P_n(t) + \lambda(n-1)P_{n-1}(t) - \mu(n-1)nP_n(t) + \mu n(n+1)P_{n+1}(t), \qquad (3)$$

i.e., a typical Markov process of simple production and quadratic disappearance. Transforming to the generating function $G(z,t) = \Sigma_n P_n(t) z^n$, and introducing the new variables $\tau = \lambda t$ and $\alpha = \mu/\lambda$, we find a second-order partial differential equation:

$$G_\tau(z,\tau) = z(z-1)(G_z(z,\tau) - \alpha G_{zz}(z,\tau)) \qquad (4)$$

with the boundary condition $G(1,\tau) = 1$ (conservation of the total probability) and the initial condition $G(z,0) = z^m$ (there are m initial gluons). It is easy to see that Eq. (4) has the steady-state solution $G(z) = \exp[(z-1)/\alpha]$, which satisfies our boundary condition and which is the well-known Poisson distribution

$$P_n = \langle n\rangle^n/n!\,\exp(-\langle n\rangle), \qquad \langle n\rangle = D = 1/\alpha\,, \qquad (5)$$

where $\langle n\rangle$ is the expectation value of the number of gluons, and $D = \langle n^2\rangle - \langle n\rangle^2$ is the variance of the distribution. A steady-state solution can be found directly from Eq. (3). Setting the left side equal to zero, and writing a version of (3) for each value $n = 0, 1, 2, \ldots$, we find

$$P_n = P_1 (\lambda/\mu)^{n-1}/n!\,,$$

which, along with the condition $\Sigma_n P_n = 1$, leads to distribution (5).

An expansion of the general solution of Eq. (4) in a series in the parameter $l = 1/\tau$ again leads to a Poisson distribution in the multiplicity. In place of (4) we find

$$l^2 G_l(z,l) = -z(z-1)(G_z(z,l) - \alpha G_{zz}(z,l)). \qquad (6)$$

We seek $G(z,l)$ as a series $G(z,l) = \Sigma_n \sigma_n(z) l^n/n!$; for the first terms of the expansion we find

$$\sigma_0(z) = a\alpha \exp(z/\alpha) + b, \qquad \sigma_1(z) = c\alpha \exp(z/\alpha) + d$$

(a, b, c, and d are arbitrary constants). In the limit $l \to 0$ ($t \to \infty$) the system thus tends asymptotically toward a steady-state Poisson distribution (this assertion could also be proved more rigorously). This result is in sharp distinction from the negative binomial distribution which results from simple gluon production, $g \to gg$, alone [$\alpha = 0$ in (4)] (Ref. 3):

$$G(z, \tau) = z^m / (z - (z-1)\exp(\tau))^m ,$$

$$\langle n \rangle = m \exp(\tau), \qquad D = \langle n^2 \rangle / m - \langle n \rangle.$$

A negative binomial distribution successfully describes the experimental data on the multiplicity distribution of the charged particles in hadron-hadron interactions at today's energies,[4] $\sqrt{s} = 20\text{–}900$ GeV.

Consequently, a narrowing of the multiplicity distribution of the observable particles (i.e., of the gluons which reach the hadronization mass Q_0) will thus signal that the gluon recombination has come into play at high energies: a transition from a negative binomial distribution with a large variance $D \sim \langle n \rangle^2$ to a narrow Poisson distribution with a variance $D \sim \langle n \rangle$. The value of $\langle n \rangle$ of the steady-state distribution which is established will be equal to $1/\alpha$, the ratio of the probabilities of the simple production and quadratic disappearance of gluons.

A time-varying solution of Eq. (4) and an analysis of the case with t-dependent values of λ and μ will be published separately.

I wish to thank V. V. Bazhanov, O. L. Zorin, A. K. Likhoded, and Yu. G. Stroganov for many constructive discussions.

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Translated by Dave Parsons