Zeeman effect in a charge-monopole system

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The Zeeman effect in a weak field in a system with a spinless charged particle and a monopole will be quite different from the splitting of the levels of a particle which has a spin. This result contradicts the identification of such a system as a fermion.

There has recently been an active discussion in the literature of the possibility of constructing a fermion from bosons for topologically nontrivial entities in field theory. As possible examples, the papers cited below have considered composite systems including spinless particles and Dirac and 't Hooft-Polyakov monopoles. In the present letter we show that it is incorrect to identify such systems as fermions.

A spinless charged particle in the field of a Dirac monopole has a dynamic integral of motion. This integral takes its simplest form in the global space of a Kustanheim-Stifel stratification $\mathbb{C}^2\setminus 0$ with a base $\mathbb{R}^3\setminus 0$ and a layer U(1), to which the dynamics of the particle can be transferred.[1] Corresponding to this integral are the generators of the SU(2) group:

$$J_i = \frac{1}{2}\,(\bar{z}\sigma_i\bar{\partial} - \partial\sigma_i z), \qquad i = 1, 2, 3, \tag{1}$$

where $z\in\mathbb{C}^2\setminus 0$ and $\bar{z}\sigma_i z = x_i \in\mathbb{R}^3\setminus 0$. The angular part of the wave function is an element of the rotation matrix $D^j\,(n/2)m(\xi)$, $\xi\in S^3$ (Ref. 2), where $n = 2eg$ is an integer; e is the charge of the particle; g is the charge of the monopole; $m = -j, -j+1,..., j$; and $j = n/2,\ n/2+1, (n/2)+2,...$ (we are assuming $n>0$).

It is asserted in several papers (see Refs. 3–5 and the bibliographies there) that the interaction of a particle with a Dirac monopole would result in the generation of a spin of $n/2$ in the system, and the addition of numbers 1, 2,... would correspond to orbital excitations. In particular, if n were odd, a fermion could form from two bosons: a spinless particle and a spinless Dirac monopole.

In this connection it is interesting to check just how a system of this sort would behave in a weak uniform magnetic field. Taking terms linear in the field **H** into account, we can write the Hamiltonian $\mathscr{H}$ in a global stratification space as

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H} = -\frac{1}{2M}\left(\frac{1}{\bar{z}z}\,\partial\bar{\partial} + \frac{v^2}{4(\bar{z}z)^2}\right) + V - \frac{eH_i}{2M}\left(\frac{\bar{z}\sigma_i z}{\bar{z}z}\,\frac{v}{2} + J_i\right),$$
$$v = z\partial - \bar{z}\bar{\partial}, \tag{2}$$

where V is a potential which localizes the charged particle. We are assuming that the position of the monopole is fixed. For a dyon, for example, we would have $V = -\alpha/$

$\bar{z}z$, and the energy levels of the discrete spectrum of the unperturbed Hamiltonian $\mathscr{H}_0$ would be nondegenerate with respect to j.

It is then obvious that a splitting of levels is given by the matrix elements

$$\Delta E = -\frac{eH}{2M} < njm \mid J_3 + \frac{\bar{z}\sigma_3 z}{\bar{z}z}\frac{v}{2} \mid njm > , \tag{3}$$

where $|njm\rangle = R_j^{(n)}(\bar{z}z)\sqrt{(2j+1/4\pi)}D^j(n/2)m(\xi)$, $v|njm\rangle = n|njm\rangle$, and $\mathbf{H} = H\mathbf{e}_3$. The integral in (3) is over the global stratification space with a volume element $dV = (zz/\pi)dz^1dz^2d\bar{z}^1d\bar{z}^2$.

After an integration over a 3D sphere of the product of three D functions

$$D^{*j}_{-(n/2)m} D^j_{-(n/2)m} D^1_{00} ,$$

where $D^1_{00} = (\bar{z}\sigma_3 z/\bar{z}z)$, we find

$$\Delta E = -\frac{eHm}{2T}\left(1 - \frac{n^2}{4j(j+1)}\right). \tag{4}$$

Let us consider the lowest orbital state with $j = n/2$. If we were dealing with an actual spin, we could use the well-known formula

$$\Delta E = -\frac{e}{2M} 2mH, \qquad -\frac{n}{2} \leqslant m \leqslant \frac{n}{2}, \tag{5}$$

for the magnetic sublevels. Actually, it can be seen from (4) that we have

$$\Delta E = -\frac{e}{2M} m H\left(\frac{n}{2} + 1\right)^{-1} \tag{6}$$

in this case. The difference between (5) and (6) is completely understandable when we note that the dynamic-symmetry operators J_i bear no relation to the rotation generators in rotations of the coordinate system in a transformation to a base $\mathrm{R}^3\setminus 0$. This circumstance is also manifested, in particular, in a change in the phase of the wave function because the system spends time in the magnetic field: Over a period corresponding to the Larmor precession of the spin, with the frequency μH, the wave function of our system acquires a phase factor $\exp\{2\pi i n/2n + 3\}$, while in the case of a true spin $n/2$ this factor would be $(-1)^n$. It is clear that the generators (1) of the dynamic-symmetry group cannot be interpreted directly as spin operators.

The same can be said of a system consisting of a 't Hooft-Polyakov monopole which is interacting with a scalar isodoublet. In a long series of studies, beginning with Refs. 6 and 7, it has been asserted that although all the original fields are boson fields, a half-integer spin will appear in a system because of the existence of the integral

$$\mathbf{J} = \mathbf{L} + \mathbf{T}, \tag{7}$$

where $\mathbf{L} = [\mathbf{r}\times\mathbf{p}]$ and $\mathbf{T} = \vec{\sigma}/2$ is an isospin operator. Since the stratification is trivial, it is convenient to carry out the analysis in ordinary space in this case.

In a weak uniform magnetic field, we add to the Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2M}[p_j - \kappa A_j^a T_a]^2 + V,$$

where $\kappa A_j^a = \epsilon_{ajk}[(1 - h(r))/r^2]x_k$ is the potential of a 't Hooft-Polyakov monopole, the terms

$$\Delta\mathcal{H} = -\frac{eH}{2M}\Big\{\Big(T_z + \frac{1}{2}\Big)L_z + \frac{1}{2}(1 - h(r))\Big[\Big(T_z + \frac{1}{2}\Big)\sin^2\theta$$

$$-\frac{1}{2}\cos\theta\,(T_+(n_x - in_y) + T_-(n_x + in_y))\Big]\Big\}, \qquad \mathbf{n} = \mathbf{r}/r, \tag{8}$$

which determine a level splitting

$$\Delta E = -\frac{eH}{2M}\,\frac{l \pm m + 1/2}{2l + 1}\Bigg(m - \frac{1}{2} + \frac{(1 - \overline{h(r)})}{4\big(l - \frac{1}{2}\big)\big(l + \frac{3}{2}\big)}$$

$$\times\Big[\Big(l - \frac{1}{2}\Big)\Big(l + \frac{3}{2}\Big) + m^2 - m\Big(l \mp m - \frac{1}{2}\Big)\Big]\Bigg), \qquad j = l \pm \frac{1}{2}. \tag{9}$$

Here we find $\overline{h(r)}$, the expectation value over the radial functions $R_{jl}(r)$. Consequently, again in this system, the Zeeman effect is anomalous in an arbitrarily weak field, and the generators of the dynamic-symmetry group in (1) are totally unrelated to the spin operators which perform the transformation of the wave functions of the system during rotations.

We wish to emphasize that the transformation properties of the wave fields under transformations of the frame of reference determine whether the energy will be of fixed sign and thus determine the quantum statistics of the particles corresponding to the fields.

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