Energy spectrum of bubbles in a liquid

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A model of an ideal incompressible liquid is used to show that the energy spectrum of bubbles is quantized. The momentum distribution function of bubbles is determined. In liquid He^4 the bubble spectrum coincides with the "multibackground" branch of the elementary excitations.

Let us consider a bubble in an ideal incompressible liquid of density ρ, which has a surface tension σ and which is at the saturation line far from the critical point where the vapor density is low in comparison with the density of the liquid. The oscillatory motion of the liquid is quantized and the spherically symmetric motion of the bubble walls has the smallest quantum of motion. At low temperatures we can therefore restrict the analysis to one oscillatory degree of freedom which corresponds to radial motion. The rotary motion is also quantized and the null quantum corresponds to the absence of rotation. Since we are restricting the analysis to low temperatures, we will ignore this quantum. Because of the translational motion of the bubble and the vibration of its walls, the motion of the liquid is irrotational. The total energy of the liquid containing the bubble of radius R is[1]

$$E = \int (\rho v^2/2) d^3r + 4\pi\sigma R^2 = E_T + E_R ,$$
$$E_T = 3P_T^2/4\pi\rho R^3 , \qquad E_R = P_R^2/8\pi\rho R^3 + 4\pi\sigma R^2 , \tag{1}$$

where $\mathbf{v}(\mathbf{r})$ is the velocity of the liquid, and E_T, E_R and P_T, P_R are the energy and momentum of the translational and radial motions ($P_R = 4\pi\rho R^3\dot{R}$).

The translational motion of a bubble is a classical motion and its radial motion is a quantized motion. The dispersion law can be determined from the Bohr-Sommerfeld quantization conditions, $\oint P_R dR = 2\pi\hbar(n + 1/2)$. Using (1), we find the equation for determining $E_n(P_T)$

$$\tilde{E}_n(\tilde{P}_T) = F(\tilde{P}_T^2/\tilde{E}_n^{5/2}), \tag{2}$$

where $\tilde{E}_n(x) = E_n(x)/E(0)$, $\tilde{P}_T = P_T/P_n$, and

$$E_n(0) = \frac{2^{4/7}\pi\hbar^{4/7}\sigma^{5/7}\left(n + \frac{1}{2}\right)^{4/7}}{I^{4/7}(0)\,\rho^{2/7}}, \quad P_n^2 = \frac{\rho E_n^{5/2}(0)}{6\pi^{1/2}\sigma^{3/2}} . \tag{3}$$

The function $F(y) = [I(0)/I(y)]^{4/7}$ is defined by the expressions

$$I(y) = \int_{x_1(y)}^{x_2(y)} (x^3 - x^5 - y)^{1/2}\, dx, \qquad I(0) = \frac{\Gamma(1/2)\Gamma(1/4)}{2i\,\Gamma(3/4)} \approx 0.25, \tag{4}$$

where $x_{1,2}$ are positive nulls of the integrand ($x_1 \leqslant x_2$). A spectrum similar to (3) for the case $P_T = 0$ was obtained by Lifshitz and Kagan[2] for the energy of quantum nuclei. For real liquids this energy is comparable to the heat of vaporization of the atom. At $T = 1.1$ K, for example, $E_0(0) \approx 15.5$ K for He^4 and at $T = 4.2$ K, $E_0(0) \approx 6.5$ K. At the triple point of water $E_0(0) \approx 410$ K.

In the entire region of allowable values of y the function $I(y)$ is approximated, within 2%, by the expression

$$I(y) \approx I(0)[1 - (y/y_0)^{3/4}], \quad y_0 = 2 \times 3^{3/2}/5^{5/2} \approx 0.19. \tag{5}$$

We thus establish a relationship between P_T and E_n

$$\tilde{P}_T^2 \approx y_0 \tilde{E}_n^{5/2} [1 - \tilde{E}_n^{-7/4}]^{4/3}. \tag{6}$$

If the momenta are low, we have $\tilde{E}_n \approx 1 + P_T^{3/2} + \dots$. The quantity E_n depends on P_T in a nonquadratic manner because of the oscillation of the additional mass of the bubble $[M_b \sim R^3(t)]$ and of its velocity.

Figure 1 is a plot of relation (6) in ^4He at $T = 1.1$ K for the first two modes. Also shown in this figure is the dispersion law for an ideal phonon gas ($E = cP_T$, where c is

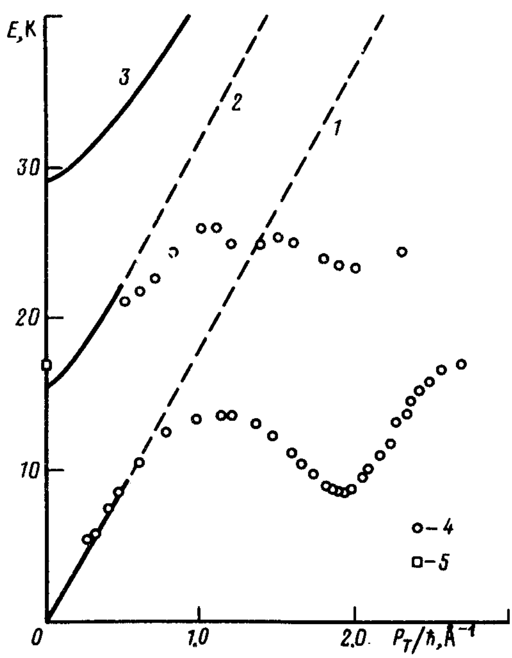

FIG. 1. Energy spectrum of single-particle excitations in liquid He^4 at $T = 1.1$ K. Ideal gas of phonons (1) and of ground-state bubbles (2) and bubbles in the first excited state (3). Points—Experiment of Ref. 3 (4); experiment of Ref. 4 (5).

the velocity of sound) and the results obtained in the experiments on inelastic neutron scattering[3] and Raman scattering of light.[4] The high-frequency branch of $E(P_T)$, usually called the "multibackground" branch,[5] in our view, determines the bubble spectrum. The dashed part of curve 2 differs from the experiment because we ignored the interaction between quasiparticles, whose role increases with increasing velocity. The fact that bubbles disappear when two or more phonons are released limits their lifetime. This is apparently one of the reasons for a marked broadening of the multibackground branch of the spectrum which was observed experimentally.[3,5]

The maximum size of a bubble at rest $R_n = [E_n(0)/4\pi\sigma]^{1/2}$ increases with increasing n. At large values of n it is proportional to $n^{2/7}$. The size of the bubble in the ground state is close to the atomic spacing in the liquid (in He^4 $R_0 \approx 2.2$ Å at $T = 1.1$ K) and the fair agreement with the experiment in the region of small values of P_T, which we obtained on the basis of the hydrodynamic model, should not be attributed too much significance. The ground-state bubbles more likely resemble the vacancies in the crystals.

Bubbles constitute one type of collective excitations in a liquid and we will call them bablons in the ensuing discussion. As in the case of vacancies in crystals,[6] the statistical base of bablons is determined by the number of atoms that must be destroyed during their formation by the atomic spin. Since, in general, $E_n > E_n(0) \gg T$, in the nonideal-gas approximation the equilibrium distribution of bablons is given by the Boltzmann equation with a zero chemical potential,

$$n(P_T) = \sum_n \exp(-E_n(P_T)/T). \tag{7}$$

The bablon concentration can be estimated by integrating Eq. (7) over the momenta, with allowance for the fact that the principal contribution comes from the small values of P_T:

$$N \approx A \frac{\rho T^2}{\hbar^2 \sigma} \sum_n (n + {}^1/_2) \exp(-E_n(0)/T), \tag{8}$$

where $A \approx 7.1 \times 10^{-3}$. An estimate for He^4 at $T = 1.1$ K gives $N \approx 2.4 \times 10^{13}$ cm^{-3} and at $T = T_\lambda$ it gives $N \approx 2.5 \times 10^{17}$ cm^{-3}.

The interaction between bablons is attributable principally to their radial fluctuations which produce a velocity field around the bablon, whose potential is equivalent to the electrostatic potential of the charge $Z = 2(\pi\rho)^{1/2} \times R^2 \dot{R}$. Since $\langle Z \rangle = 0$, the asymptotic behavior of the pair interaction between bablons, $V(r)$, is of a polarization nature.

$$V(r) = -\alpha/2r^4, \qquad \alpha = \langle Z_1 R_2^3 + Z_2 R_1^3 \rangle / 2. \tag{9}$$

The specular-reflection forces cause the bablons to be repelled from the container walls, making their presence in the wall regions, in thin films and in thin capillaries less desirable.

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