Representation of the propagator of a Dirac particle in an external gravitational field as a sum over paths

A. V. Marshakov and V. Ya. Faĭnberg
P. N. Lebedev Physics Institute, Academy of Sciences of the USSR

(Submitted 2 March 1988)
Pis'ma Zh. Eksp. Teor. Fiz. **47**, No. 10, 481–484 (25 May 1988)

A representation of a Green's function of the Dirac equation in a curved space as a sum over paths is derived on the basis of locally supersymmetric generalization of the action of a Dirac particle in an external gravitational field.

In the present letter we will fully substantiate the representations of the Green's function of a Dirac particle in a gravitational field, which satisfies a first-order equation. The representation of the propagator of a Dirac particle in a curved space as a path integral has a long history, whose beginning is traceable to the work of De Witt.[2] References to the subsequent studies may be found, for example, in the article by De Alfaro *et al.*[3] We begin by considering the action of a Dirac particle in an external gravitational field, which was proposed by Brink *et al.*[1] This action can easily be written in the reparametrization-invariant and locally supersymemtric form

$$S = \int_0^1 d\tau \, \frac{1}{2} \{ g_{\mu\nu}(x(\tau))[e^{-1} \dot{x}^{\mu} \dot{x}^{\nu} - i \psi^{\mu} \dot{\psi}^{\nu} - i e^{-1} \chi \dot{x}^{\mu} \psi^{\nu}] - i\Gamma_{\mu,\nu\lambda}(x(\tau)) \psi^{\mu} \psi^{\nu} \dot{x}^{\lambda} \} ;$$

$$\mu, \nu, \lambda = 0, 1, ..., D-1 \quad \dot{x}^{\mu} = dx^{\mu}/d\tau, \ \Gamma_{\mu,\nu\lambda} = \frac{1}{2}(\partial_{\nu} g_{\mu\lambda} + \partial_{\lambda} g_{\mu\nu} - \partial_{\mu} g_{\nu\lambda}), \tag{1}$$

$$\partial_{\mu} = \partial/\partial x^{\mu},$$

where ψ^{μ} and χ are Grassmann variables.

The invariance of (1) under reparametrization is self-evident; its invariance under local supersymmetry was proved elsewhere.[6] On the basis of action (1), the path representation of a Dirac particle has the form

$$\int De \int DX \int_{x(0)=x_0}^{x(1)=x_1} D_g x \int D_g \psi \, e^{-S(x^{\mu}, \psi^{\mu}; g_{\mu\nu})}, \tag{2}$$

where S is action defined by Eq. (1) (in Euclidean terms).

The measure of integration over the variables $x^{\mu}(t)$ in a curved space-time depends on the external metric $g_{\mu\nu}(x)$ in the following way[7] ($g = \det g_{\mu\nu}$):

$$D_g x \sim [g(x_0) g(x_1)]^{-1/4} [\prod_t \sqrt{g(x(t))}] Dx, \tag{3}$$

where Dx is a measure of integration in a flat space. In evaluating the path integral over the Grassmann variables in a curved space we must assume

$$D_g \psi \sim \left[\prod_t \frac{1}{\sqrt{g(x(t))}}\right] D\psi \ , \tag{4}$$

where $D\psi$ is a measure of integration in a flat space. Definition (4) is a corollary of the rules of integration over the anticommuting variables:

$$\int d\psi_\mu \psi^\nu = \delta^\nu_\mu \, ,$$

according to which the differential $d\psi_\mu$ has a subscript.

Despite the fact that action (1) depends exclusively on the external metric $g_{\mu\nu}(x)$, the Green's function evaluated in (2) depends on the vierbein fields $V_{a\mu}(x)$. The vierbein-field dependence should manifest itself when the boundary conditions for the fields $\psi^\mu(t)$ are determined if we require that $\psi^\mu \sim \gamma^\mu$, and the vierbein field is always included in the evaluation of the γ^μ matrices in a curved space: $\gamma^\mu = V^\mu_a(x)\gamma^a$. Because of the circumstances indicated above, it is more difficult to determine directly in (2) the boundary conditions for the fields $\psi^\mu(t)$. Accordingly, we will change the variables in (2):

$$\psi^\mu = V^\mu_a \psi^a, \qquad g_{\mu\nu}(x) = V_{a\mu}(x) V^a_\nu(x).$$

Action (1) then becomes

$$S = \int_0^1 d\tau \frac{1}{2}\{g_{\mu\nu} e^{-1} \dot{x}^\mu \dot{x}^\nu - i\psi_a \dot{\psi}^a - i(V_{a\nu}\partial_\lambda V^\nu_b$$

$$+ \Gamma_{\mu,\nu\lambda} V^\mu_a V^\nu_b)\psi^a \psi^b \dot{x}^\lambda - ie^{-1}\chi \dot{x}^\mu V_{a\mu}\psi^a\} \tag{5}$$

and the propagator is defined as follows:

$$\hat{G}(1,0\,|\,V) = \int De\, D\chi\, D_g x\, D\psi\, e^{-S(x^\mu,\, \psi_a;\, V_{a\mu})} \tag{6}$$

$$x(0) = x_0, \qquad x(1) = x_1; \qquad \psi_a(t) = \gamma_a + \phi_a(t),$$

where $D_g x$ is defined in (3), and $D\psi$ is a measure of integration over the fields $\psi_a(t)$. The boundary conditions, which are satisfied by the variables $\psi_a(t)$, are defined just as they are in a flat space.[5] The standard boundary conditions for the variables $x^\mu(t)$ are used to determine the propagator: the specified values at the path ends.

To fully substantiate (6), we evaluate the expression

$$\hat{j}_{a\mu}(1,0\,|\,x) = \frac{\delta \hat{G}(1,0\,|\,V)}{\delta V_{a\mu}(x)}\bigg|_{V_{a\mu}(x) = \delta_{a\mu}} . \tag{7}$$

Variation of (7) in expression (6) should take into account the vierbein-field dependence of not only action (5) but also the expression for the measure of integration in (3), which can be rewritten in the form

$$D_g x \sim [g(x_0)g(x_1)]^{-1/4}\, e^{(1/2)\delta(0)\int dt \log g(x(t))}\, Dx.$$

For a one-dimensional metric $e(t)$ and a gravitino $\chi(t)$ we use the standard gauge

$$e(t) = T, \qquad \chi(t) = X.$$

Integrating over X, we will then find for expression (7)

$$\hat{j}_{a\mu}(1,0|x) = \int_0^\infty dT \int_0^1 d\tau \Big\langle \Big\{ \frac{(x_1 - x_0)_\lambda \langle \psi_\lambda \rangle_\psi}{2T} \Big(\delta_{a\mu}\, \delta(0) - \frac{\dot{x}_a(\tau)\dot{x}_\mu(\tau)}{T} \Big)$$

$$- \frac{1}{4T}\int_0^1 dt\, \dot{x}_\lambda(t) [\dot{x}_\nu(\tau) \langle \psi_\lambda(t)\, \psi_\mu(\tau)\, \psi_a(\tau)\rangle_\psi + \dot{x}_\mu(\tau)\langle \psi_\lambda(t)\psi_a(\tau)\psi_\nu(\tau)\rangle_\psi$$

$$+ \dot{x}_a(\tau)\langle \psi_\lambda(t)\, \psi_\mu(\tau)\, \psi_\nu(\tau)\rangle_\psi]\, \partial_\nu + \frac{\dot{x}_\mu(\tau)\langle \psi_a \rangle_\psi}{2T} \Big\}\, \delta(x - x(\tau)) \Big\rangle_x$$

$$- \frac{1}{2}\delta_{a\mu}(\delta(x - x_1) + \delta(x - x_0))\, \hat{G}(1,0), \tag{8}$$

where $\hat{G}(1,0)$ is the free propagator of a Dirac particle. Averaging over $x^\mu(t)$ and $\psi_a(t)$ in (8) is understood to mean the path integral over these variables with the boundary conditions (6). Since the evaluation is now carried out in flat space, the entities γ_μ satisfy (after the evaluation of the integral) the standard anticommutation relations:

$$[\gamma_\mu, \gamma_\nu]_+ = -\delta_{\mu\nu}.$$

Integration over the fields $\psi_a(t)$ in (8) is carried out in accordance with the following rules:

$$\langle \psi_\lambda(t)\rangle_\psi = \gamma_\lambda, \qquad \langle \psi_\lambda(t)\, \psi_a(\tau)\, \psi_\nu(\tau)\rangle_\psi$$

$$= \frac{1}{2}(\gamma_\lambda \gamma_a \gamma_\nu - \gamma_\nu \gamma_a \gamma_\lambda) + \frac{1}{2}\epsilon(t-\tau)(\gamma_\nu \delta_{a\lambda} - \gamma_a \delta_{\nu\lambda}). \tag{9}$$

The singularities $\sim\delta(0)$ in (8) cancel out exactly [this can be shown by means of a discrete representation of the path integral over the variables $x^\mu(t)$]. Evaluating (8) by means of Eq. (9), we find

$$\hat{j}_{a\mu}(1,0|x) = -\frac{1}{2(2\pi)^{D/2}}\{(\partial_0 - \partial_1)_\mu \not{\partial}_1 \gamma_a \not{\partial}_0 + a \leftrightarrow \mu\}\, G_{1x} G_{x0}$$

$$+ \frac{1}{2}\not{\partial}_1 G_{1x}\gamma_\mu \gamma_a \delta_{x0} + \frac{1}{4}\delta_{\mu a}\not{\partial}_1 G_{1x}\delta_{x0} + \frac{1}{2}\delta_{1x}\gamma_a\gamma_\mu \not{\partial}_x G_{x0}$$

$$+ \frac{1}{4}\delta_{a\mu}\delta_{1x}\not{\partial}_x G_{x0} - \frac{1}{2}\delta_{a\mu}(\delta_{1x} + \delta_{x0})\hat{G}_{10}, \tag{10}$$

where

$$\Box_x G_{xy} = -(2\pi)^{D/2}\delta_{xy}, \qquad \delta_{xy} = \delta(x-y)$$

$$\Box = \partial_\mu \partial_\mu , \qquad \not{\partial} = \gamma_\mu \partial_\mu .$$

Switching in (10) to the standard normalization, we finally find

$$\hat{j}_{a\mu}(1,0|x) = -\frac{i}{\sqrt{2}}\{-\frac{1}{4}\left(\hat{J}_{1x}\Gamma_a \overleftrightarrow{\partial}_\mu \hat{J}_{x0} + \hat{J}_{1x}\Gamma_\mu \overleftrightarrow{\partial}_a \hat{J}_{x0}\right)$$

$$+\frac{1}{4}(\hat{J}_{1x}\Gamma_\mu\Gamma_a\delta_{x0} + \delta_{1x}\Gamma_a\Gamma_\mu\hat{J}_{x0}) - \frac{3}{4}\left(\hat{J}_{1x}\delta_{x0} + \delta_{1x}\hat{J}_{x0}\right)\}, \qquad (11)$$

where

$$[\Gamma_\mu, \Gamma_\nu]_+ = 2\delta_{\mu\nu}, \qquad \Gamma_\lambda \partial^x_\lambda \hat{J}_{xy} = -\delta_{xy} .$$

Result (11) is equal, within a small factor, to the expansion, with respect to the deviation of the vierbein field $V_{a\mu}$ from $\delta_{a\mu}$, of the first-order equation for the Green's function of a Dirac particle in an external gravitational field. The results of the evaluation will be published later. A representation of the Green's function of a Dirac particle in external electromagnetic and non-Abelian gauge fields in a first-order equation was derived in Refs. 4–6.

We wish to thank A. A. Tseĭtlin for a discussion of the features of the measure of integration in a curved space.

[1]L. Brink, P. Di Vecchia, and P. Howe, Nucl. Phys. **B118**, 76 (1977).
[2]B. S. De Witt, Phys. Rev. **85**, 653 (1952); Rev. Mod. Phys. **29**, 377 (1957).
[3]V. De Alfaro, S. Fubini, G. Furlan, and M. Roncadelli, Nucl. Phys. **B296**, 402 (1988); Preprint CERN-TH. 4849/87.
[4]V. V. Marshakov and V. Ya. Faĭnberg, Pis'ma Zh. Eksp. Teor. Fiz. **46**, 253 (1987) [JETP Lett. **46**, 319 (1987)].
[5]V. Ya. Faĭnberg and A. V. Marshakov, P. N. Lebedev Physics Institute, Preprint No. 338, 1987; Nucl. Phys. B (in press).
[6]A. V. Marshakov and V. Ya. Faĭnberg, FIAN Rapid Communications, Physics, No. 3, 1988.
[7]M. Mizrahi, J.Math. Phys. **16**, 2201 (1975).

Translated by S. J. Amoretty