Chiral bosonization on Reimann surfaces in an operator formalism

A. M. Semikhatov
P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow

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An operator algebra of a fermion *bc* system on a Riemann surface *C* is represented globally in terms of operators of a conformal boson theory on *C*. The operator relations generate bosonization formulas for correlation functions. The construction is based on an operator generalization of the Baker-Akhiezer function.

One of the basic examples of a conformal theory on Riemann surfaces is given by a system of J- and $(1-J)$-differentials b and c (Ref. 1),[1)] an bosonization is apparently the most effective way to work with a *bc* system.[2–5] The results of bosonization are expressions for the correlation functions of the *bc* system, which are interpreted as the correlation functions of some boson theory of the scalar-field type. Both in path-integral approaches[2–4] and when a τ function is used,[5] however, just which boson conformal theory on the Riemann surface corresponds to the *bc* system remains unknown. In the case of bosonization with the help of a τ function, the use of this function as an intermediate step obscures the question of the global construction for the boson operators corresponding to fermions. In the path-integral approach, on the

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other hand, the boson theory is poorly defined since the global properties of the Riemann surface are dealt with through the introduction of a multivalued scalar field φ which has a local "quantum" part and a "classical" part, which allows (and even prescribes) the occurrence of jumps as nontrivial cycles are traced out.

Our purpose in the present letter is to construct a systematic operator bosonization of a (fermion) *bc* system. In other words, we wish to outline a conformal boson theory on a Riemann surface and a system of operators of this theory which globally represent the operators of a *bc* system. This construction essentially reduces to the realization of the three following comments: a) The scalar field $\varphi(z)$ which participates in local bosonization formulas,[1]

$$\hat{b}(u) = e^{-\varphi(u)}, \quad \hat{c}(u) = e^{\varphi(u)}, \qquad e^{-\varphi(u)} e^{\varphi(v)} = \frac{1}{v-u} : e^{-\varphi(u)} e^{\varphi(v)} : \quad (1)$$

($\hat{}$ means bosonization), and which is not a genuine conformal field, even locally,[1] must be regarded as a linear integral $\int_z \partial\varphi$ of the current $\partial\varphi$. In case (1), the latter is the same as the current j of the *bc* system, which is (we wish to stress this point) globally defined on a surface of an arbitrary type. b) The jumps in the field φ can now be interpreted in a natural way as the result of a coiling of the integration contour around a homology. c) An object which generalizes $\exp \int^z j$ but which is insensitive to the coiling of the contour is the Baker-Akhiezer function.[7] The operator-valued Baker-Akhiezer functions perform a global bosonization [see Eqs. (3), (4), and (7) below].

1. The primary entity of the boson theory is thus the current $\tilde{I}$: the operator 1-differential on Riemann surface C of type g. Normalizing to zero a periods, we introduce the current $I = \tilde{I} - \Sigma_{j=1}^{g} \omega_j (1/2\pi i) \oint_{a_j} \tilde{I}$, where ω_j are holomorphic 1-differentials.[8] In the equations that follow, the normal ordering of the constitutent operators constructed from the current I is generally reached by subtracting all possible pairings; here

$$\overbrace{I(u)I(v)} \equiv \langle 0 | I(u) I(v) | 0 \rangle = d_u d_v \ln E(u, v) , \quad (2)$$

where E is the principal form of the surface,[8] and $|0>$ is the vacuum in the boson theory.

2. The bosonization procedure should incorporate the ghost-number balance theorem, according to which the only nonvanishing *bc* correlation functions are those in which there are $L \equiv (2J-1)(g-1)$ more appearances of b than c. We can correspondingly suggest bosonization formulas only for operators which are (differential) polynomials in b and c with the specified predominance of the operators b.

In the simplest case we have only the product $b(P_1)...b(P_L), P_\alpha \in C$. We fix a point of common position $Q \in C$. We construct an operator $B_J(P_\alpha, Q)$ such that the following holds for correlation functions of C:

$$\langle \mathcal{O} \rangle_C = \langle 0 | \hat{\mathcal{O}} B_J | 0 \rangle .$$

We assume that $\mathcal{P}$ represents the divisor $\mathcal{P} = \Sigma P_\alpha$, while Δ is the Riemann class.[8] We can then write

$$B_J = :\exp\left(\frac{2J-1}{2\pi i}\sum_{j=1}^{g} \oint_{a_j}\omega_j(y)\int_Q^y I\right)$$

$$\times\exp\left(-\int_{(2J-1)\Delta}^{\mathcal{P}} I\right)\theta\left(\mathcal{P}-(2J-1)\Delta-\oint_b I\right): \qquad (3)$$

We will show that the agrument of the the θ function in (3) can be written in more detail as

$$\int_{(2J-1)\Delta}^{\mathcal{P}}\omega_i - \oint_{b_i} I = \sum_{\alpha=1}^{L}\int_R^{P_\alpha}\omega_i - (2J-1)\kappa_i^{(R)} - \oint_{b_i} I,$$

where $\kappa^{(R)}$ is the vector of Riemann constants. In (3) and in the equations below, we are also assuming θ functions with characteristics determined by the spinor structure of the bc theory.[4] The first exponential factor in (3) specifies the background charge[1] $Q = 2J-1$; the other factors determine the operator L-point Baker-Akhiezer function.

3. All that can appear in the correlation functions $\langle b...bc...c\rangle$ against the background of $b(P_1)...b(P_L)$ is a certain number of pairs $b(u)c(v)$. For this product (but not for the b and c operators separately!) we have

$$\widehat{b(u)c(v)} = \frac{1}{E(u,v)} : \exp\int_u^v I : \left(\frac{E(v,Q)}{E(u,Q)}\right)^{(2J-1)g}\frac{E(u,(2J-1)\Delta)}{E(v,(2J-1)\Delta)}, \qquad (4)$$

where for the divisor $\mathscr{A} = \Sigma P_i - \Sigma Q_i$ we have written

$$E(u,\mathscr{A}) = \prod_i E(u,P_i)\left(\prod_j E(u,Q_j)\right)^{-1}.$$

The product of all the principal forms in (4) is a J-differential with respect to u and a $(1-J)$ differential with respect to v. We might note that expression (4) is a generalization of local formulas (1), which can be rewritten as

$$\widehat{b(u)c(v)} = \frac{1}{v-u} : \exp\int_u^v \partial\varphi:$$

On the right side of (4) we do not have a θ function of the type $\theta(\oint_b I + ...)$ which compensates for the dependence of the integral on the coiling, in contrast with the Baker-Akhiezer function in (3). An attempt to introduce it explicitly in (4) would be unacceptable since it would prevent us from obtaining the expressions found in Ref. 4 for the correlation functions. The situation can be improved remarkably, however, by means of the following result regarding the normal ordering:

$$:\exp\int_{\mathscr{A}}^{\mathscr{B}} I::\theta\left(e-\oint_b I\right): = :\theta\left(e+\mathscr{A}-\mathscr{B}-\oint_b I\right)\exp\int_{\mathscr{A}}^{\mathscr{B}} I: \qquad (5)$$

($\mathscr{A}$ and $\mathscr{B}$ are divisors of degree n). Using this expression along with another, similar result,

$$:\exp\int_u^v I::\exp\frac{2J-1}{2\pi i}\sum_{j=1}^{g}\oint_{a_j}\omega_j(y)\int_Q^y I:$$

$$=\left(\frac{\sigma(u)}{\sigma(v)}\right)^{2J-1}\left(\frac{E(u,Q)}{E(v,Q)}\right)^{(2J-1)g}\cdot(\text{norm. order}), \quad (6)$$

where

$$\sigma(u)=\exp\left(-\frac{1}{2\pi i}\sum_{i=1}^{g}\oint_{a_i}\omega_i(y)\ln E(y,u)\right),$$

we find

$$\widehat{b(u)c(v)B_J}=\frac{1}{E(u,v)}\left(\frac{\sigma(u)}{\sigma(v)}\right)^{2J-1}\frac{E(u,\mathscr{P})}{E(v,\mathscr{P})}:\exp\int_u^v I\exp-\int_{(2J-1)\Delta}^{\mathscr{P}} I$$

$$\exp\frac{2J-1}{2\pi i}\sum_{i=1}^{g}\oint_{a_i}\omega_i(y)\int_Q^y I\quad\frac{\theta(\mathscr{P}-(2J-1)\Delta+u-v-\oint_b I)}{\theta(\mathscr{P}-(2J-1)\Delta)}:. \quad (7)$$

The effect of (5) is to shift the argument of the θ function in (7) by an amount $u-v=-\int_u^v\omega_k$, which is exactly what we need in order to reconstruct the Baker-Akhiezer structure and thus to compensate for the coiling $\exp\oint_b I$. Furthermore, a "tuning" of the argument of the θ function occurs in a similar way for any number of pairs $b(u_i)c(v_i)$ on the left side of (7). On the right side of (7), yet other factors appear as a result of the normal ordering. They turn out to be of precisely such a nature that the correct expressions for the correlation functions are reproduced when we take the expectation value $\langle 0|\ldots|0\rangle$. In particular, we find directly from (7)

$$\langle 0|\widehat{b(u)c(v)}B_J|0\rangle=\frac{1}{E(u,v)}\left(\frac{\sigma(u)}{\sigma(v)}\right)^{2J-1}$$

$$\times\frac{E(u,\mathscr{P})}{E(v,\mathscr{P})}\quad\frac{\theta(u-v+\mathscr{P}-(2J-1)\Delta)}{\theta(\mathscr{P}-(2J-1)\Delta)},$$

which is the same as the expression for the bc propagator on a Riemann surface.[4]

4. Some significant simplifications arise in the case $J=1/2$. In particular, comparing (3) with the equations of Ref. 9, we find the interesting operator identity predicted by A. O. Radul:

$$\exp\left(\frac{1}{2\pi i}\right)^2\oint_Q b(u)\oint_Q c(v)S(u,v)=\text{const}\,\theta\left(-\oint_b I\right),$$

where $S(u,v)$ is the regularized propagator for spin-1/2 fermions.[9,4,8] Furthermore, for an arbitrary value of J we find from (3) the expression of Ref. 5 for an algebra-geometry τ function.

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[1)]In the case $J=2$ we find ghosts in the boson string: a source of "applied" interest in *bc* systems. String requirements have also attracted interest in Riemann surfaces of type$\geqslant 2$ (Refs. 2–6).

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