Hawking temperature and fluctuations of a scalar field in a de Sitter space

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The diffusion equation for fluctuations of a scalar field in a de Sitter space is shown to correspond to the Hawking temperature. The relationship between the steady-state solution of this equation and a Hartle–Hawking instanton is established for a space of arbitrary dimensionality and for an arbitrary type of gravitational action.

Starobinskiĭ[1] has shown that the behavior of vacuum fluctuations of a scalar field with a wavelength greater than the dimensions of the horizon in a de Sitter space is described by a diffusion equation

$$\frac{\partial \rho(\varphi,t)}{\partial t} = \frac{H_0^3}{8\pi}\frac{\partial^2 \rho}{\partial \varphi^2} + \frac{1}{3H_0}\frac{\partial}{\partial \varphi}\left(\rho\frac{dV(\varphi)}{d\varphi}\right), \tag{1}$$

where H_0 is the Hubble constant, $V(\varphi)$ is the potential of the scalar field, and $\rho(\varphi,t)$, is a corresponding probability density. In this letter we are offering a derivation of Eq. (1) which, while not completely rigorous, is extremely simple from the technical standpoint. Furthermore, it relates this equation to the Hawking temperature $T = H_0/2\pi$, which is characteristic of a de Sitter space.

As we know (Ref. 2, for example), a detector in a de Sitter vacuum and at rest in the coordinates of a closed de Sitter space detects an emission with a temperature T of quanta of the field φ. We choose a spectral characteristic for the detector such that the detector responds only to quanta having a frequency $\omega \ll H^{-1}$. Such a detector will then be coupled with only the long-wave fluctuations of the field φ ($\lambda \gg H^{-1}$), which are of a classical nature, as Starobinskiĭ[1] has pointed out. In other words, the detector is at a thermal equilibrium with uniform fluctuations of the scalar field in the region which is causally related to it. These fluctuations are therefore of a Brownian-motion nature at the temperature T. The diffusion equation for Brownian particles in a potential $U(x)$ at a temperature θ is

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\theta}{\gamma}\frac{\partial^2 \rho}{\partial x^2} + \frac{1}{\gamma}\frac{\partial}{\partial x}\left(\rho\frac{dU(x)}{dx}\right), \tag{2}$$

where γ is a viscosity coefficient. In a three-dimensional space, we would have $\gamma = 3H_0$ for a scalar field. Since $U(x)$ corresponds to the energy of the scalar field inside the volume which is causally related to the detector, $V_3 = (4/3)\pi H^{-3}$, when we replace $U(x)$ by $V(\varphi)$ we must simultaneously replace θ by T/V_3. Since we have $T = H_0/2\pi$, we obtain Eq. (1). The diffusion coefficient calculated by Kofman and Starobinsky[3] for a scalar field for the case of an n-dimensional space agrees with the use in place of V_3 of the volume of an n-dimensional sphere, $V_n = \pi^{n/2}/\Gamma[(n/$

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2) + 1]H^n. If inflation is sustained by the field φ itself, the Hubble constant H is determined by $V(\varphi)$. The exponential part of the steady-state distribution $\rho(\varphi)$is found from the equation

$$-j = \frac{T}{nHV_n}\frac{d\rho}{d\varphi} + \frac{\rho}{nH}\frac{dV}{d\varphi} = 0 \tag{3}$$

and is of the form

$$\ln\rho = C - \int T^{-1} V_n(H(\varphi))dV(\varphi) . \tag{4}$$

We relate the right side of (4) to the action I for a Hartle–Hawking instanton[4] (a relationship of this sort has been established by Starobinskiĭ[1] for a three-dimensional space and for the Einstein gravitational action; a generalization of this result is given by Pollock[5]). Since this instanton is a Euclidean sphere of radius R, we have $I(R) = [F(R) - V(\varphi)]S_{n+1}(R)$, where the function $F(R)$ depends on the specific gravitational Lagrangian, and $S_{n+1}(R) = 2\pi^{(n/2)+1}\times\Gamma^{-1}[(n/2)+1]R^{n+1}$ is the area of an $(n+1)$-dimensional sphere. We replace the variable R by S_{n+1}; we then have $I(S_{n+1}) = [F(S_{n+1}) - V(\varphi)]S_{n+1}$. The value of S_{n+1} is found from the principle of least action:

$$I'_{S_{n+1}} = (FS_{n+1})'_{S_{n+1}} - V(\varphi) = 0. \tag{5}$$

Since we have $R = H^{-1}$ and $T = H/2\pi$, we have $S_{n+1}(R) = V_n(H)T^{-1}$. We can thus evaluate the integral on the riqht side of(4).

$$\int V_n T^{-1} dV(\varphi) = \int S_{n+1}\, dV(\varphi)$$
$$= S_{n+1} V(\varphi) - \int V(\varphi) dS_{n+1} = S_{n+1} V(\varphi) - FS_{n+1} = -I .$$

These calculations reflect the properties of a Legendre transformation for the gravitational action from the variable S_{n+1} to $V(\varphi)$.

In summary, for an arbitrary Lagrangian of a gravitational field, and for a space of arbitrary dimensionality, the exponential part of the steady-state solution of Eq. (1)is of the form $\rho = e^I$ where I is the action for a Hartle–Hawking instanton.

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[1]A. A. Starobinskiĭ, Fundamental Interactions, MGPI, Moscow, 1984; A. A. Starobinsky, in: Field Theory, Quantum Gravity and Strings (H. J. de Vega and N. Sánchez, editors), Lecture Notes in Physics, Vol. 246, Springer, Berlin, 1986, p. 107; A. S. Goncharov and A. D. Linde, Fiz. Elem. Chastits At. Yadra **17**, 837 (1986) [Sov. J. Part. Nucl. **17**, 369 (1986)].

[2]N. Birrel and P. Davis, Quantum Fields in Curved Space, Cambridge Univ. Press, Oxford, 1982 (Russ. transl., Mir, Moscow, 1984),

[3]L. Kofman and A. Starobinsky, Phys. Lett. **188B**, 399 (1987).

[4]J. B. Hartle and S. W. Hawking, Phys. Rev. **D28**, 2960 (1983).

[5]M. D. Pollock, Nucl. Phys. **298B**, 673 (1988).

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