The spin topological index of ^{3}He

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A domain wall in ^{3}He-*B* is used to show that the quasiparticle spectrum depends on the spin. A modified topological index, which implies that there is a "coupling" of energy with the spin, is shown to exist.

The topological properties of the quasiparticle spectrum of the superfluid phases of ^{3}He have recently attracted increased interest.[1–4] The complexity of the order parameter of a superfluid ^{3}He gives rise to many mechanisms for various topological perturbations (domain walls, vortices, etc.) which in turn manifest themselves in the topology of the quasiparticle spectrum.

In the case of a microscopic description in terms of the Bogolyubov operator, it is a nontrivial problem to determine the quasiparticle spectrum if the order parameter has a complex texture. It would therefore be reasonable to deal with simple characteristics of the spectrum from which direct physical conclusions could be drawn. In the case of textures in the A phase,[1] for example, the presence of a gapless branch of the spectrum gives rise to the existence of unpaired fermions at $T=0$. One such characteristic is the invariant η. Specifically, in determining the spectrum

$$H_{\mathrm{B}}\Psi = E\Psi \,, \tag{1}$$

where H_{B} is the Bogolyubov operator, and Ψ is the Bogolyubov spinor, we can introduce a crude topological characteristic of the spectrum—the spectral asymmetry of the Hamiltonian H_{B} or otherwise its index:

$$\eta\,[H_{\mathrm{B}}] = \sum_{E>0} 1 - \sum_{E<0} 1 \quad . \tag{2}$$

The index shows the extent to which the levels with a positive energy differ from those with a negative energy [Eq. (2) in this case should be regularized accordingly]. Balatskiĭ and Konyshev[2] showed that $\eta\neq 0$ directly implies some important physical consequences (the normal component, uncompensated current, etc.). In considering the A phase, however, the spectrum was assumed to be spin degenerate. In general, this situation is not encountered. The domain walls in the B phase, the A and B phase boundaries, and the cores of various vortices in a superfluid ^{3}He are some examples. We will show here in the particular case of a domain wall in the B phase that the spin degeneracy is lifted in the quasiparticle spectrum. The index η in this case must be determined accurately, with allowance for the spin degree of freedom.

Let us consider a domain wall in the B phase

$$A_{\mu i} = \begin{pmatrix} 1 & & 0 \\ & \tanh x/\lambda & \\ 0 & & 1 \end{pmatrix} . \tag{3}$$

The order parameter used in (3), which is similar to the numerical solution in Ref. 4, is useful in an analytical study. For this order parameter the Bogolyubov operator can be written in the form[2]

$$H_{\mathrm{B}} = \tau_3\left(\frac{p^2}{2} + \tilde{\epsilon}\right) + \alpha\tau_1\,(\sigma_x p_z - \sigma_z p) + \alpha\tau_2 p_y\, iT\partial_p \; . \tag{4}$$

In this case we have linearized $\tanh x/\lambda$, $T=\lambda^{-1}$ in the neighborhood of $x=0$ and we switched to a momentum representation, $\tilde{\epsilon}=\dfrac{p_z^2+p_y^2}{2}-\mu$, $\alpha=\Delta_0/k_F$, Δ_0 is the gap modulus, τ_i is a Pauli matrix in the particle space (a hole), and σ_i is a Pauli matrix in the spin space. After a unitary transformation: $\tilde{H}=U^{-1}HU$, $\tilde{\Psi}=U\Psi$

$$U = \exp\,(i\tau_2\,\pi/4 + i\varphi/2\sigma_y), \qquad \tan\varphi = -\,p_z/p_0 . \tag{5a}$$

We have

$$\tilde{H} = \tau_3 \alpha \sigma_z (p_0^2 + p_z^2)^{1/2} - \tau_2\left(\frac{p_0^2}{2} + \tilde{\epsilon} + p_0 m\right) + \alpha \tau_2 T p_y i\partial_m \ . \tag{5b}$$

Here we have expanded the coordinate $p \approx p_0 + m$, where $p_0 = \pm\sqrt{-2\tilde{\epsilon}}$ (the approximate solution is described in more detail in Ref. 2). Finally, we find the anomalous level of the spectrum.

$$E_0 = \alpha S_z (2\mu - p_y^2)^{1/2} \operatorname{sgn}(T p_y p_0)$$

$$\tilde{\Psi}_{n=0} = \begin{pmatrix} \psi_0 \\ 0 \\ 0 \end{pmatrix} \exp\left(-\frac{p_0 m^2}{2\alpha T p_y}\right)\theta\,(p_0/Tp_y) + \begin{pmatrix} 0 \\ 0 \\ \psi_0 \end{pmatrix} \exp\left(\frac{p_0 m^2}{2\alpha T p_y}\right)\theta\,(-p_0/Tp_y), \tag{6}$$

$$S_z \Psi_0 = \hat{\sigma}_z \Psi_0 \ .$$

Here sgn x and θ are the sign function and the step function. The anomalous level (6) which we have found clearly breaks the symmetry $p_y \to -p_y$ and depends on the projection of the spin S_z (in transformed coordinates). We thus immediately see that the spin is transported along the y axis, i.e., the spin current $j_y^{\sigma_z}$ flows along the domain wall.[1)] The gap in the spectrum vanishes at $p_y = \pm k_F$. This approximation does not allow us to calculate the state density $N(E=0)$, since the expansion which we are using $p \approx p_0 + m$ is incorrect in the limit $E \to 0$ ($p_y \to \pm k_F\ \tilde{\epsilon} \to 0, p_0 \to 0$). This situation does not, however, prevent us from drawing important conclusions concerning the topology of the spectrum.

We move on directly to the calculation of the index $\eta[H]$. Since the anomalous branch of spectrum (6) has been found, we can write η in a specific representation. It is easy to see that the expression for index (2) requires a supplementary definition, since there is a discrete parameter S_z which changes the energy of the zeroth-level sign. Using (2), we obtain $\eta = 0$, an incorrect result, since the contributions from the branches with $S_z = \pm 1$ cancel each other. For an analytic calculation one can use the formula[2]

$$\eta[H] = \frac{2}{\sqrt{\pi}} \operatorname{Tr} \int_0^\infty dy\, e^{-y^2 H^2} H = \sum_n \frac{E_n}{|E_n|}, \tag{7}$$

where Tr is the trace of the complete system of functions and of the spins. In the case involving spectrum (6), Eq. (7) leads to a trivial result, $\eta = 0$, because of the spin dependence of the energy mentioned above. This contradiction can be easily reconciled by pointing out that in (7) the projection operator $P_\pm$, which makes us use the trace of the functions with a specific spin value, should be incorporated into the trace in (7). In our case $P_\pm = (1 \pm \sigma_z)/2$. As a result, we have

$$\eta_\pm = \frac{2}{\sqrt{\pi}} \operatorname{Tr} \int_0^\infty dy\, P_\pm H e^{-y^2 H^2} P_\pm = \pm \operatorname{sgn} T p_y\, \theta(-\tilde{\epsilon}). \tag{8}$$

This expression implies that the asymmetry of the spectrum depends on the spin projection, as the approximate solution has shown.

We have shown, therefore, that the quasiparticle spectrum of systems with complex order parameters (the example of the *B* phase was used here) depends on the spin. The nontrivial spin dependence, i.e., a dependence on a certain discrete parameter of the problem, is given by a modified index in which the operator projecting onto the states with a definite spin must be taken into account. We have also shown that the spin degeneracy (the Kramers degeneracy) is lifted in the absence of a magnetic field. The system remains invariant under the time reversal in the presence of a domain wall. The coupling of the spin with a quasiparticle momentum occurs without a spin-orbit coupling because of the properties of the uniform order parameter: in the *B* phase the spin angular momentum cancels the orbital angular momentum. Similar phenomena presumably occur in other objects: vortex cores, interfaces of *A*-*B* phases.

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[1)]In Ref. 4, the energy was also found to depend on the spin in another kind of domain wall.

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