Four-loop correction to $\sigma_{tot}(e^+e^- \to \text{hadrons})$ in QCD and the problem of determining the parameter $\Lambda_{\overline{MS}}$

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Calculations of the four-loop correction of order $O(\alpha_s^3)$ to $\sigma_{tot}(e^+e^-$ hadrons) in QCD are reported. There is a discussion of how the large correction found influences the procedure for determining the value of the parameter $\Lambda_{\overline{MS}}$ from data from e^+e^- colliders in various energy ranges.

In recent years the e^+e^- annihilation into hadrons has been regarded as one of the cleanest areas from the theoretical standpoint for tests of quantum chromodynamics (QCD). An important characteristic of this annihilation is the ratio $R(s) = \alpha_{tot}(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$, on which we now have experimental information over the wide energy range $0.25\ \text{GeV} \lesssim \sqrt{s} \lesssim 50\ \text{GeV}$, and for which we expect to see precise measurements at the Z^0-boson peak. Analysis of the data by means of the finite-energy sum rules,[1] the Borel sum rules[3,4] which were introduced in Ref. 2, a

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comparison of $R_{\text{theo}}(s) \approx R_{\text{expt}}(s)$ above the thresholds for b-quark production,[5,6] and other methods has raised the hope that it would be possible to extract reliable information on the magnitude of the QCD parameter $\Lambda_{\overline{MS}}$. Since the three-loop perturbation-theory corrections $O(\alpha_s^2)$ to $R(s)$ have turned out to be small,[7] it has seemed that there is no reason to doubt the applicability of perturbation theory for describing the behavior $R(s)$.

In this letter we would like to briefly discuss some problems which have arisen in the analytic calculation of the next correction, $O(\alpha_s^3)$. A direct calculation has been carried out in the Euclidean region $Q^2 = -q^2 > 0$ for the function

$$D(Q^2) = Q^2 \int_0^\infty \frac{R(s)}{(s+Q^2)^2}\, ds. \tag{1}$$

Incorporating effects of the analytic continuation of $D(Q^2)$ into the physical region leads to an additional term proportional to π^2 in $R(s)$ in the order of perturbation theory in which we are interested:

$$R(s) = D(s) - 3\Sigma Q_f^2 \pi^2 \frac{\beta_0^2}{3}\left(\frac{\alpha_s}{\pi}\right)^3, \tag{2}$$

where β_0 is the first coefficient of the QCD β function, for which a three-loop expression is known in the MS scheme[8]:

$$\frac{1}{\pi}\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) = -\beta_0\left(\frac{\alpha_s}{\pi}\right)^2 - \beta_1\left(\frac{\alpha_s}{\pi}\right)^3 - \beta_2\left(\frac{\alpha_s}{\pi}\right)^4,$$

$$\beta_0 = \left(11 - \frac{2}{3}f\right)\frac{1}{4}, \qquad \beta_1 = \left(102 - \frac{38}{3}f\right)\frac{1}{16}, \tag{3}$$

$$\beta_2 = \left(\frac{2857}{2} - \frac{5033}{18}f + \frac{325}{54}f^2\right)\frac{1}{64}.$$

Using the correction $(O\alpha_s^3)$, we can write the renormalized expression for $R(s)$ as

$$R(s) = 3\Sigma Q_f^2 \left\{ 1 + \frac{\alpha_s}{\pi} + \left(a_1 - a_2 \ln\frac{s}{\mu^2}\right)\left(\frac{\alpha_s}{\pi}\right)^2 + \left(b_1 - b_2 \ln\frac{s}{\mu^2} + b_3 \ln^2\frac{s}{\mu^2}\right)\left(\frac{\alpha_s}{\pi}\right)^3 \right\} - (\Sigma Q_f)^2 c_1 \left(\frac{\alpha_s}{\pi}\right)^3. \tag{4}$$

In the $\overline{MS}$ scheme, the three-loop correction $O(\alpha_s^2)$is not large[7]: $a_1 = 1.986 - 0.115f$. To find the correction $O(\alpha_s^3)$, it was necessary to calculate the contributions of more than 100 four-loop diagrams. Calculations have been carried out with the help of SCHOONSCHIP,[9] a special program written in the language of an analytic-calculation system, which implements an algorithm of integration by parts.[10] Here are the numerical values of the four-loop coefficients which we found in the $\overline{MS}$ scheme:

$$b_1 = 70{,}985 - 1{,}200f - 0{,}005f^2 , \quad c_1 = 1{,}679 , \tag{5}$$

The coefficients of the logarithmic terms are $a_2 = 2.75 - 0.167f$, $b_2 = 17.298 - 2.086f + 0.038f^2$ and $b_3 = 7.562 - 0.917f + 0.028f^2$. They can also be found from the renormalization-group relations $a_2 = \beta_0$, $b_2 = \beta_1 + 2\beta_0 a_1$ and $b_3 = \beta_0^2$. A test of their validity has served as a test of the correctness of the calculations. Applying the renormalization group to (4) is equivalent to setting the logarithmic terms equal to zero and making the substitution $\alpha_s \to \bar{\alpha}_s$. The running coupling constant can be expressed in terms of $L = \ln(s/\Lambda^2_{\overline{MS}})$:

$$\frac{\bar{\alpha}_s}{\pi} = \frac{1}{\beta_0 L} - \frac{\beta_1 \ln L}{\beta_0^3 L^2} + \frac{1}{\beta_0^5 L^3} (\beta_1^2 \ln^2 L - \beta_1^2 \ln L + \beta_2 \beta_0 - \beta_1^2). \tag{6}$$

To study the effect of the large correction which we calculated, (5), on the value of the parameter $\Lambda_{\overline{MS}}$, we will use the result $\bar{\alpha}_s(34^2 \text{ GeV}^2) = 0.145 \pm 0.020$, which was recently found through a comparison of the three-loop approximation for $R(s)$ in the $\overline{MS}$ scheme with data from the PETRA and PEP accelerators.[6] For the three-loop approximation, using the first two terms in (6), we find $\Lambda_{\overline{MS}} = 280^{+230}_{-165}$ MeV. Incorporating four-loop correction (5) and the complete expression (6), we find the result $\bar{\alpha}_s(34^2 \text{ GeV}^2) = 0.132^{+0.013}_{-0.016}$ and an approximately twofold (!) decrease in the parameter $\Lambda_{\overline{MS}}$ $\Lambda_{\overline{MS}} = 155^{+125}_{-85}$ MeV. We wish to stress that the agreement of this estimate with the values found for $\Lambda_{\overline{MS}}$ from other processes[11] can be considered a demonstration of a successful use of perturbation theory in QCD only if the higher-order corrections to $R(s)$ and to the characteristics of other processes are not larger than the effects which have been taken into account.

Let us examine the perturbation-theory series for our case. For the value $\bar{\alpha} \approx 0.132$ and $f = 5$, the expression for (R,s) in the $\overline{MS}$ scheme is[2)]

$$R(s) = \frac{11}{3}(1 + 0.042 + 0.0025 + 0.0048) \tag{7}$$

The last term taken into account is seen to be roughly twice the preceding term.

There is no doubt that perturbation-theory series are asymptotic. It is believed that their accuracy is determined by the first discarded term. There is the possibility that a minimum correction is present among the unknown terms in (7). In such a case, the incorporation of the calculated correction in the analysis of the data available from the PEP, PETRA, and TRISTAN accelerators and also future data from the SLC and LEP colliders would be correct. If, however, the third term in (7) turns out to be the minimum term, then a perturbation-theory approximation containing only the first two terms of the series will be the most accurate. Its use will not make it possible to correctly determine the value of the parameter $\Lambda_{\overline{MS}}$, since the terms $O(\alpha_s^2)$, which fix the scheme, are discarded in this process.

When we go to lower energies (i.e., to larger values of $\bar{\alpha}_s$), the incorporation of the higher-order perturbation-theory corrections becomes even more problematical. To illustrate the point, we consider the analysis of low-energy data from e^+e^- colliders on the basis of the Borel sum rules[2]:

$$M_n = \frac{1}{M^2} \int_0^\infty R(s)\, e^{-s/M^2} \left(\frac{s}{M^2}\right)^n ds. \quad (8)$$

In the case $f = 3$, the perturbation-theory contributions to the first two sum rules are

$$M_0 = 1 + \frac{\bar{\alpha}_s}{\pi} + \left(\frac{\bar{\alpha}_s}{\pi}\right)^2 2{,}939 + \left(\frac{\bar{\alpha}_s}{\pi}\right)^3 83{,}919 \; ,$$
$$M_1 = 1 + \frac{\bar{\alpha}_s}{\pi} + \left(\frac{\bar{\alpha}_s}{\pi}\right)^2 0{.}690 + \left(\frac{\bar{\alpha}_s}{\pi}\right)^3 66{,}698 \quad (9)$$

where $\bar{\alpha}_s = \bar{\alpha}_s(M^2)$. In the region $M^2 \sim 1$ GeV^2, where the fit was made in Refs. 3 and 4, the perturbation-theory series explodes. Specifically, the corrections $O(\bar{\alpha}_s^3)$ in (9) are comparable in magnitude to the leading corrections $O(\bar{\alpha}_s)$ for the value $\bar{\alpha}_s \approx 0.38$, which was reached in the analysis of the Refs. 3 and 4. The estimates $\Lambda_{\overline{MS}} = 100$–$200$ MeV which were found in Refs. 3 and 4 thus cannot be regarded as reliable. In such cases, it seems highly likely that using only the first two terms in series (9) would be justified. However, as we have already mentioned, that approach does not lead to a correct determination of the value of the parameter $\Lambda_{\overline{MS}}$, since the terms which fix the scheme are discarded in this approach.

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[2)]In the derivation of (7), the suppression of the last term in (4) by a factor $(\Sigma Q_f)^2/3\Sigma Q_f^2 = 1/33$ for $f = 5$ was taken into consideration.

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