Extension of the Penrose representation and its use to describe supersymmetric models

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An extension of the Penrose representation is proposed [Eq. (4)]. It is shown through the use of this extension that massless superparticles (superstrings) and spin particles (fermion strings) in 4-space can be described in a common fashion in terms of commuting Weyl spinors $u_A(\tau)$ $(u^{\alpha}{}_A(\tau,\sigma))$ and Grassmann Weyl spinors $\theta_A(\tau)$ $(\theta_A(\tau,\sigma))$, which are free of the constraints generated by local (super) symmetries.

Reaching an understanding of the mechanism for the cancellation of anomalies in superstrings[1] means establishing clear constraints between global and local supersymmetries of Green-Schwarz superstrings and the sheet supersymmetry of Neve-Schwarz fermion strings.[2] In this letter, the existence of such constraints, even at the level of the actions of the spin[3] and supersymmetric[4] particles, is established. They correspond to the limit of an infinite string tension. In addition, it is proved that they are equivalent on the mass shell of the 4-space spinors $u_A(\tau)$.

The action of a massless particle with a spin of 1/2 in 4-space,[3]

$$S_{1/2} = \int d\tau \left[p^{\dot{A}A}\dot{x}_{A\dot{A}} - \frac{i}{2}\psi^{\dot{A}A}\dot{\psi}_{A\dot{A}} - \frac{e}{2}p^{\dot{A}A}p_{A\dot{A}} - \frac{i}{2}\chi p^{\dot{A}A}\psi_{A\dot{A}}\right], \quad (1)$$

is characterized by two constraints, which generate the Klein-Gordon and Dirac equations,

$$p^{\dot{A}A}p_{A\dot{A}} = 0\,, \qquad p^{\dot{A}A}\psi_{A\dot{A}} = 0\,. \quad (2)$$

To describe particles with a zero spin and mass, Penrose[5] introduced a spinor representation for their 4-momenta $P_{A\dot{A}} = u_A\bar{u}\dot{A}$, which automatically allows the constraint $p^2 = 0$, by virtue of the relation $u^A u_A = 0$. In this paper we generalize Penrose's approach to particles with a spin, and we introduce some auxiliary (with respect to u_A) Grassmann spinors θ_A. It thus becomes possible to write the Grassmann 4-vector $\psi_{A\dot{A}}$ in a form which allows constraints (2) for particles with spin:

$$p_{A\dot{A}} = u_A\bar{u}_{\dot{A}}\,, \qquad \psi_{A\dot{A}} = u_A\bar{\theta}_{\dot{A}} + \theta_A\bar{u}_{\dot{A}} \quad (3)$$

Action (3) and the equations of motion for the variables u_A, θ_A are

$$S_{1/2} = \int d\tau \left[u^A\bar{u}^{\dot{A}}\dot{x}_{A\dot{A}} - \frac{i}{2}(u^A\bar{\theta}^{\dot{A}} + \theta^A\bar{u}^{\dot{A}})(u_A\bar{\theta}_{\dot{A}} + \theta_A\bar{u}_{\dot{A}})^{\cdot}\right], \quad (4)$$

a) $\dot{u}_A = i\beta(\tau)u_A(\tau)$, b) $\dot{\theta}_A = \frac{1}{4}\chi(\tau)u_A(\tau)$. (5)

Representation (4) of the action is invariant under transformations of a local sheet supersymmetry with a real parameter $\alpha(\tau)$,

$$\delta u_A = 0, \qquad \delta\theta_A = \frac{1}{2}\alpha u_A , \qquad \delta x_{A\dot{A}} = i\alpha(u_A\bar{\theta}_{\dot{A}} + \theta_A\bar{u}_{\dot{A}}), \tag{6}$$

and also under $U(1)$ local transformations, $\delta u_A = iau_A, \delta\theta_A = ia\theta_A$, and transformations of the auxiliary real local supersymmetry $\delta\theta_A = i\mu u_A, \delta u_A = 0$ which do not alter representations (3). Using the $U(1)$ symmetry and the equations of motion for u_A in (5a), we can convert action (4) into the action of a supersymmetric particle[4,6] with $p_{A\dot{A}} = u_A\bar{u}_{\dot{A}}$:

$$S'_{1/2} = \int d\tau u^A\bar{u}^{\dot{A}}[\dot{x}_{A\dot{A}} - \frac{i}{2}(\dot{\theta}_A\bar{\theta}_{\dot{A}} - \theta_A\dot{\bar{\theta}}_{\dot{A}})] = S_{\rm sp} . \tag{7}$$

This action is invariant under transformations of the global supersymmetry,

$$\delta u_A = 0, \quad \delta\theta_A = \epsilon_A , \quad \delta x_{A\dot{A}} = \frac{i}{2}(\theta_A\bar{\epsilon}_{\dot{A}} - \epsilon_A\bar{\theta}_{\dot{A}}), \tag{8}$$

and of the local Siegel supersymmetry,[7] which in this approach is found through a complexification of $\alpha(\tau)$ of local supersymmetry (6):

$$\delta u_A = 0, \quad \delta\theta_A = 2\alpha u_A , \quad \delta x_{A\dot{A}} = i\alpha(\theta_A\bar{u}_{\dot{A}} + u_A\bar{\theta}_{\dot{A}}). \tag{9}$$

Making use of the equality of numbers of degrees of freedom of the supersymmetric and spinor particles, we reach the conclusion that they are classically equivalent when the equations of motion for the spinors $u_A, \bar{u}_{\dot{A}}$ in (5) hold. It follows from these equations that the evolution of $u_A, \bar{u}_{\dot{A}}$ reduces to $U(1)$ gauge transformations.

Representation (3) is a simple consequence of the condition of an inverse Higgs effect[8] for the locally [(6)] and globally [(8)] supersymmetric form $W_{\bar{\eta}A\dot{A}}$, constructed from the superfields $X_{A\dot{A}}(\tau,\eta)$ and $\Theta_A(\tau,\eta)$:

$$W_{\bar{\eta}A\dot{A}} = D_{\bar{\eta}}X_{A\dot{A}} - i(D_{\bar{\eta}}\Theta_A\bar{\Theta}_{\dot{A}} + \Theta_A D_{\bar{\eta}}\bar{\Theta}_{\dot{A}}), \tag{10}$$

$$X_{A\dot{A}}(\tau,\eta) = x_{A\dot{A}}(\tau) + i\eta e^{1/2}\psi_{A\dot{A}}(\tau) ,$$

$$\Theta_A(\tau,\eta) = \sqrt{2}(\theta_A(\tau) + \frac{1}{2}\eta e^{1/2}u_A(\tau)),$$

where $D_{\bar{\eta}} = E^{-1}(\tau,\eta)(\partial_\eta + i\eta\partial_\tau)$. In component form, we find from (10)

$$\psi_{A\dot{A}} = u_A\bar{\theta}_{\dot{A}} + \theta_A\bar{u}_{\dot{A}} , \qquad \dot{x}_{A\dot{A}} = u_A\bar{u}_{\dot{A}} + 2i(\dot{\theta}_A\bar{\theta}_{\dot{A}} - \theta_A\dot{\bar{\theta}}_{\dot{A}}). \tag{11}$$

The Penrose representation for $P_{A\dot{A}}$ along with equation of motion (5b), is found from

(11) after the substitution $\dot{x} = ep - (i/2)\chi\psi$ and the solution of constraint (2).

The twistor approach under consideration here can be generalized to strings. As in the case of particles, the constraint between the Grassmann variables of a fermion string[2] and a superstring[1] (or heteroidal string) is established with the help of the condition for an inverse Higgs effect for the sheet spinor form $W_{\alpha A\dot{A}}$ (for $N = 2$ supersymmetry, $i = 1, 2$):

$$
\begin{aligned}
W_{\alpha A\dot{A}} &= D_\alpha X_{A\dot{A}} - i(D_\alpha \Theta^i_A \bar{\Theta}^i_{\dot{A}} + \Theta^i_A D_\alpha \bar{\Theta}^i_{\dot{A}}),\\
X_{A\dot{A}} &= x_{A\dot{A}}(\xi^m) + i\eta^\alpha \psi_{\alpha A\dot{A}}(\xi^m) + \frac{i}{2}(\eta^\alpha\eta_\alpha) F_{A\dot{A}}(\xi^m),\\
\Theta^i_A &= \theta^i_A(\xi^m) + \eta^\alpha u^i_{\alpha A}(\xi^m) + \frac{i}{2}(\eta^\alpha\eta_\alpha)\rho^i_A(\xi^m).
\end{aligned}
\tag{12}
$$

In a superconformal gauge, with $D_\alpha = E^{-1/2}\left(\frac{\partial}{\partial\eta^\alpha} - i(\eta\gamma^m)_\alpha \frac{\partial}{\partial\xi^m}\right)$, we find a representation for $\partial_m x_{A\dot{A}}$ and the string spinor Grassmann field $\psi_{\alpha A\dot{A}}$ in terms of the spinor fields of a superstring, θ^i_A and u^i_A from the condition for an inverse Higgs effect, (12):

$$
\begin{aligned}
\psi_{\alpha A\dot{A}} &= u^i_{\alpha A}\bar{\theta}^i_{\dot{A}} + \theta^i_A \bar{u}^i_{\alpha\dot{A}}\\
\partial_m x_{A\dot{A}} &= i(\partial_m \theta^i_A \bar{\theta}^i_{\dot{A}} - \theta^i_A \partial_m \bar{\theta}^i_{\dot{A}}) - (\bar{u}^i_{\dot{A}}\gamma_m u^i_A),
\end{aligned}
\tag{13}
$$

and also the equation of motion for the spinors θ^i_A and a representation of $F_{A\dot{A}}$ in terms of ρ^i_A and u^i_A Armed with the constraint between the Grassmann variables of a superstring and a fermion string, we can determine the constraint of their Lagrangians, as in the case of particles, discussed above.

We would like to point out yet another natural generalization which follows from the twistor approach. Using the locally and globally supersymmetric forms $W_{\bar{\eta}A\dot{A}}$, in (10) for particles or in (12) for strings, one can construct doubly supersymmetric superfield actions for particles and strings. For particles, this action generalizes the action of spin superparticles and takes the form

$$
S = \frac{-i}{2}\int d\tau d\eta E(\tau,\eta)[c_1 W^{\dot{A}A}_{\bar{\tau}} W_{\bar{\eta}A\dot{A}} + c_2 D_{\bar{\eta}} X^{\dot{A}A} D_{\bar{\eta}}\Theta_A D_{\bar{\eta}}\bar{\Theta}_{\dot{A}}]. \tag{14}
$$

An analogous locally and globally supersymetric action for strings can be found from the action of a fermion string[2] by replacing the planar covariant derivative $D_\alpha X_{A\dot{A}}$ by invariant form (12), with the possible appearance of a Weiss-Zumino term, characteristic of strings with a global supersymmetry:

$$
S = \int d^2\xi d^2\eta [c_1 W^{\alpha\dot{A}A} W_{\alpha A\dot{A}} + c_2 (W-Z\text{-term})]. \tag{15}
$$

This approach can be generalized in a natural way to spaces of dimensionality $D = 6$ and 10, where there is a profound relationship with quaternion and octanion algebras with division[11] and twistors for $D = 10$ (Ref. 10).

After this work had been completed, D. Sorokin informed us that actions (14) and (15) are being studied independently by An. Kavalov and R. Mkrtchyan.

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