Linear wave conversion in a plasma without a hybrid resonance

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(Submitted 19 April 1988)
Pis'ma Zh. Eksp. Teor. Fiz. **48**, No. 2, 71–74 (25 July 1988)

A new resonant mechanism for the slowing of electrostatic waves in a plasma with periodic variations has been observed.

A hybrid resonance is an interesting effect in the electrodynamics of plasmas and also one of practical importance. Mathematically, a hybrid resonance is a singularity of solutions of the equation

$$\operatorname{div}(\hat{\epsilon}\,\vec{\nabla}\varphi) = 0, \qquad (1)$$

where $\varphi(\mathbf{r})$ is a scalar potential, and $\hat{\epsilon}$ is the dielectric tensor of the cold plasma. The singularity arises because the characteristics of this equation intersect at the singular point or asymptotically approach each other.[1] In the present paper we show that a phenomenon analogous to a hybrid resonance can occur in a two-dimensionally inhomogeneous medium which is bounded along one coordinate and which is periodically inhomogeneous along the other, at frequencies at which there is no "ordinary" hybrid resonance. The most important example of this geometry is an axisymmetric toroidal configuration. Here, however, we will restrict the discussion to the most graphic example: the propagation of an oblique plasma wave in a plane plasma waveguide with a slight periodic variation along the z axis, which is the direction of the external magnetic field. This wave is described by wave equation (1), which takes the form

$$\frac{\partial}{\partial x}\,\epsilon\,\frac{\partial\varphi}{\partial x} + \frac{\partial}{\partial z}\,\eta\,\frac{\partial\varphi}{\partial z} = 0, \qquad (2)$$

where $\epsilon = \epsilon_{xx}$, and $\eta = \epsilon_{zz}$. Let us assume that ϵ and η depend on z periodically with a period of $2\pi/\kappa$ and that the relation $\epsilon > 0$ holds. The transparency region, $\eta < 0$, is confined to the interval $x_1 \leqslant x \leqslant x_2$. We consider solutions of Eq. (2) of the nature of waveguide modes which are traveling along the z axis, and we consider modes for which we can use the geometric-optics approximation. In this case the propagation of waves can be described by means of ray paths, which in this case coincide with the characteristics of Eq. (2). In a waveguide configuration, a ray path is successively reflected from the "walls" x_1 and x_2. We denote by z_i the coordinate of the point of the ith encounter of some path with a wall, and we denote by Δ_i the distance to the next such point [the $(i+1)$st]. In a medium which is inhomogeneous along z, the shift will be a function of z_i, and the dependence will obviously be periodic. At a small "modulation depth," we need retain only the first two terms of an expansion in a Fourier series: $\Delta_i(z_i) = \Delta_0 + b\sin\kappa z_i$ (we are choosing an origin for the z scale in such a way

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that there is no constant phase in the argument of the sine). The evolution of the ray path is then described by the mapping

$$z_{i+1} = z_i + \Delta_0 + b \sin \kappa z_i , \tag{3}$$

whose properties are well known.[2] Under the condition

$$| \kappa\Delta_0 - 2\pi n | < \kappa | b | , \qquad n = 1, 2 \ldots \tag{4}$$

there are two periodic paths (of such a nature that the relation $z_{i+1} = z_i + 2\pi n/\kappa$ holds), one of which is stable. All the other paths approach it exponentially. The wave vector of the wave grows in the same fashion. This behavior of the ray paths is similar to that in the case of a hybrid resonance in a plane-layer medium and with an angle between the direction of the variation and the magnetic field which is different from 0 to $\pi/2$. The existence of any asymptote for the ray paths, which are also equipotentials in the case of Eqs. (1) and (2), means that any solution of Eq. (2) will have a singularity as $z \to \infty$. The infinite slowing of the wave halts as a result of spatial dispersion, however. To analyze this case, we will put Eq. (2) in a more specific form, setting $\eta = -\eta_0(x)[1 + \beta(x)\sin \kappa z]$, and $\epsilon = 1$ and making a thermal correction in it. The corresponding dispersion relation then takes the form

$$D(\mathbf{r},\mathbf{k}) = 0, \quad D = k_x^2 - \eta_0 \left[1 + \beta(x) \sin \kappa z + 3 \frac{v_T^2 k_z^2}{\omega^2} \right] k_z^2 , \tag{5}$$

where $v_T^2 = T_e/m_e$. The ray paths are known to be the paths traced out by a mechanical system with a Hamiltonian D. To analyze them, it is convenient to make a canonical transformation from the variables x, k_x to the new variable $\alpha(x)$, which varies monotonically along the path, and to its conjugate momentum k_α, in accordance with

$$\lambda \frac{d\alpha}{dx} = \pm \sqrt{\eta_0} , \quad \lambda k_x = \pm \sqrt{\eta_0}\, k_\alpha ,$$

where λ is a constant. The old coordinate x is a periodic function of α. We choose the constant λ in such a way that the period of the function $x(\alpha)$ is 2π:

$$\lambda = \frac{1}{\pi} \int_{x_1}^{x_2} \sqrt{\eta_0}\, dx$$

In this case we have $\Delta_0 = 2\pi\lambda$, and the resonance condition $\Delta_0 \kappa = 2\pi n$ takes the form $\kappa\lambda = n$. Expanding the periodic function $\beta(x(\alpha))$ in a Fourier series, and noting that we have $z = \lambda\alpha$ in the zeroth approximation (in the small parameters β and $k_z v_T$), we can retain in the Hamiltonian the oscillatory resonant terms which are the very slowest: $\beta(\alpha)\sin \kappa z \to \beta_n \sin \zeta$, where

$$\zeta = n\alpha - \kappa z, \quad \beta_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta[x(\alpha)] \cos n\alpha\, d\alpha .$$

After this simplification, we carry out the canonical transformation $(\alpha, k_\alpha)(z, k_z)$

$\rightarrow(\alpha, p)$, (ζ, k). Here $k = k_z/\kappa$; $p = k_\alpha + nk_z/\kappa$; and the function D becomes

$$D = \left[\frac{p - nk}{\kappa\lambda}\right]^2 - k^2(1 + \beta_n \sin\zeta) - \gamma k^4 , \qquad (6)$$

where $\gamma = 3v_T^2 \kappa^2/\omega^2$. Since the Hamiltonian does not depend on α, the momentum is conserved, and the equation $D(\zeta, k, p) = 0$ makes it possible to construct phase trajectories $k(\zeta, p)$ directly. The periodic trajectory described above corresponds to the curve $p = 0$; the "cold" waves $(\gamma p^2 \ll 1)$ slow to $k_z \sim (\omega/v_T)\sqrt{\beta_n}$. The behavior of the curves can be interpreted as a linear conversion into a "warm" mode near a stable periodic path and an inverse conversion into a cold wave. The analysis above is based on the use of the geometric-optics approximation, which is difficult to defend rigorously in the case of a two-dimensionally inhomogeneous medium. We do note, however, that the conclusion that the slowing of the wave undergoes an exponential growth under condition (3) can be reached even without an analysis of the behavior of the ray paths.

In the case $\beta(x) = 0$, the field $\varphi(x,z)$ can be written as a superposition of natural waveguide modes:

$$\varphi(x, z) = \sum_m A_m \varphi_m(x) \exp(ik_m z), \qquad (7)$$

where

$$\varphi_m = \sqrt{\frac{2\lambda}{\pi}}\, \eta_0^{-1/4} \cos\left[k_m \int_{x_1}^{x} \sqrt{\eta_0}\, dx' - \frac{\pi}{4}\right], \quad k_m = \frac{m + 1/2}{\lambda}, \quad m > 0$$

This representation is also valid when a slight period modulation $|\beta| \ll 1$ is imposed on the waveguide. In this case, however, the modes interact with each other, so their amplitude varies along the length of the waveguide, $A_m = A_m(z)$. The interaction between modes should be particularly effective when the resonance conditions hold for the scattering of modes by a long-wave modulation: $k_{m+n} - k_m = \kappa$. It is easy to see that this condition is the same as condition (3) in the limit $\beta \rightarrow 0$ and holds for an arbitrary mode index m. Assuming the deviation from resonance to be small, $|k_{m+n} - k_m| \ll \kappa$, we find a system of equations describing the evolution of the amplitudes A_m. Substituting (7) into (2), and ignoring the terms which are small and those which oscillate rapidly, we find

$$\frac{\partial C_m}{\partial z} - ik_m \beta_n \delta C_m = \frac{\beta_n k_n}{4}[C_{m+n} - C_{m-n}], \qquad (8)$$

where

$$C_m = A_m k_m \exp(i\beta_n \delta k_m z), \qquad \delta = 2\left(1 - \frac{\kappa\lambda}{n}\right)\beta_n^{-1} .$$

The Fourier transformation $C_m(z) = \int_{-\infty}^{\infty} C(v,t)\exp[-(ik_m\lambda/n)v]dv$ reduces the system of recurrently coupled differential equations to the equation

$$\frac{\partial C}{\partial t} + (\sin\nu - \delta)\frac{\partial C}{\partial \nu} + \cos\nu C = 0\,, \tag{9}$$

where $t = n\beta_n z/2\lambda$. The solution of Eq. (9), $C(\nu,t)$, can be expressed in terms of the "initial condition" by the method of characteristics:

$$C(\nu, t) = C_0(\nu_0(\nu, t))\,\frac{\sin\nu_0 - \delta}{\sin\nu - \delta}\,, \qquad C_0(\nu) = C(\nu, 0)\,. \tag{10}$$

In the case $|\delta| < 1$, the characteristics of Eq. (9), $\nu_0(\nu,t)$, are given by

$$\frac{\sqrt{1-\delta^2}\cos\nu + 1 - \delta\sin\nu}{\sin\nu - \delta} = e^{-\sqrt{1-\delta^2}\,t}\,\frac{\sqrt{1-\delta^2}\cos\nu_0 + 1 - \delta\sin\nu_0}{\sin\nu_0 - \delta}\,, \tag{11}$$

and in the limit $t \to \infty$ they converge on $\nu = \nu_*$, $\cos\nu_* = -\sqrt{1-\delta^2}$. Using (10) to determine $A_m(z)$, and substituting the resulting expression into (7), we find the distribution of the electric field $E_z = \partial\varphi/\partial z$ in the waveguide:

$$E[\alpha(x), z] = E^+ + iE^-\,, \tag{12}$$

$$E^{\pm} = \frac{1}{2}\,\frac{\sin\nu_0 - \delta}{\sin\nu - \delta}\left\{E\left(\frac{\nu_0}{n},\,0\right) - iE\left(-\frac{\nu_0}{n},\,0\right)\right\}_{\nu = \kappa z \pm n\alpha}\,,$$

where $0 \leqslant \alpha \leqslant \pi$, $\varphi(\alpha + \pi k, 0)$ is determined in terms of $\varphi(\alpha,0)$ with the help of (7), and we have $\nu_0 = \nu_0(\nu,t)$ according to (11). This expression describes waves which are coming into the observation point after reflection from opposite "walls" of the waveguide and which are traveling in opposite directions in the $z = 0$ cross section. The factor in front of the braces (curly brackets) describes an exponential intensification of the field as the characteristics converge and the wave slows.

In conclusion we would like to stress that the singularity which we have been discussing here in the example of an electrostatic plasma wave is actually a characteristic of any wave system of the hyperbolic type with parameters which depend periodically on one of the coordinates or the time.

[1]A. D. Piliya and V. I. Fedorov, Zh. Eksp. Teor. Fiz. **60**, 389 (1971) [Sov. Phys. JETP **33**, 210 (1971)].
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Translated by Dave Parsons