Soliton turbulence

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A qualitative description of a strong wave turbulence in the absence of a wave collapse is given. The processes which lead to an increase in the amplitudes of the solitons as their number is reduced play the central role in the turbulence mechanism. In the conservative nonintegrable systems the soliton is a statistical attractor. The described picture is confirmed by a direct numerical simulation.

1. There is no doubt now that the development of a strong turbulence in various physical situations is accompanied by the formation of space-time structures which can be described in the coordinate space. The nonlinear Schrödinger equation is a sufficiently universal wave-turbulence model (see, e. g., Refs. 1–3):

$$i\psi_t + \Delta\psi + f(|\psi|^2)\psi = 0. \tag{1}$$

For $f(u) > 0$ and $f'(u) > cu^{(2-d)/d}$ (d is the dimensionality of space) the structures described by (1) are wave collapses, whose theory is currently being developed rapidly (see Refs. 4–6 and the literature cited there). Equation (1) has a soliton-type solution

$$\psi(\mathbf{r}, t) = g(\lambda, \vec{\xi})\exp\left[\frac{i}{2}(\mathbf{vr}) + i\left(\lambda^2 - \frac{v^2}{4}\right)t\right], \qquad \vec{\xi} = \mathbf{r} - \mathbf{v}t,$$

where the function g is a solution of the equation

$$\Delta g + f(g^2)g - \lambda^2 g = 0, \quad \nabla g|_{\xi \doteq 0} = 0, \qquad g \underset{\xi \to \infty}{\to} 0. \tag{2}$$

In the collapsed state the soliton is unstable and in the turbulent processes it does not appear at all. In contrast, if the soliton is stable [if it is of the form $sd < 4$ in the case of a power-law nonlinearity $f(u) = u^{s/2}$ (Refs. 1–3)], the solitons must play the principal role in the turbulance described by Eq. (1), i.e., the turbulence must be of a "soliton" type.[7] The nature of this turbulence is of fundamental interest. Equation (1) has integrals of motion

$$N = \int |\psi|^2 d\mathbf{r}, \quad H = \int [|\nabla\psi|^2 - \phi(|\psi|^2)]d\mathbf{r} \quad (\phi'(u) = f(u)). \tag{3}$$

Disregarding the quantum effects, the tendency of the energy toward uniform distribution based on the degrees of freedom causes the integral H (the Hamiltonian) to localize in the short-wave region. The system in this case can form a condensate: a uniform field accompanied by small-scale fluctuations. At $f'(u) > 0$, however, the unstable condensate decays into solitons in the absence of a collapse. In the special integrable case $f(u) = cu(c>0)$, the solitons scatter on each other elastically and their number is conserved. In the general nonintegrable case the blending of solitons is preferred from the thermodynamics standpoint.[2,8] As a result of blending of the solitons, a part of the integral H, which is carried away by free waves, is set free. The integral of the quasiparticle number N remains primarily in the soliton. The soliton in this case decreases in size. Accordingly, in the solution of the problem on the evolution of a given initial distribution the soliton is a kind of statistical attractor: The state decays asymptotically with respect to time into a soliton and a set of slightly nonlinear waves.

2. To directly prove this conclusion, we have numerically integrated the one-dimensional equation (1) over large evolution times for different degrees of nonlinearity of $f(u)$. The problem was solved for a fixed segment $0 \leqslant x \leqslant L$ with periodic boundary conditions and perturbed uniform initial conditions

$$\psi(x,0) = \psi_0 \left(1 + \epsilon \sin \frac{2\pi}{L} x \sin \frac{\pi\alpha}{2} x\right), \qquad \epsilon \ll 1, \quad \frac{1}{2} \leqslant \alpha \leqslant 1. \tag{4}$$

In Eq. (1) the increment of the modulational instability of the condensate $\psi_0 \exp[if(|\psi_0|^2)]$ is given by

$$\gamma(k) = k\sqrt{2A - k^2}, \qquad A = \left(u \frac{\partial f}{\partial u}\right)_{u = |\psi_0|^2} . \tag{5}$$

This increment peaks at $k = \sqrt{A}$; it is $\gamma_{\max} = A$. The corresponding modulation length, $\lambda_{\mathrm{mod}} = 2\pi/\sqrt{A}$, determines the number of solitons formed at $t > \gamma_{\max}^{-1}$, $n \approx L/\lambda_{\mathrm{mod}}$.

In the case of a power-law nonlinearity $f(u) = u^{s/2}$, the solution of Eq. (2) for $d = 1$ is

$$g(\xi) = \left(\lambda\sqrt{1 + \frac{s}{2}} \Big/ \cosh \frac{\lambda s}{2} \xi\right)^{2/s} . \tag{6}$$

In this solution we have

$$N = \frac{2\sqrt{\pi}}{s}\left(1 + \frac{s}{2}\right)^{2/s} \frac{\Gamma(2/s)}{\Gamma(2/s + 1/2)} \lambda^{(4-s)/s}, \quad H = N\left(\frac{v^2}{4} - \lambda^2 \frac{4-s}{4+s}\right). \tag{7}$$

The simulation is adequate if

$$\lambda_{mod} \ll L \ll M\lambda^{-1} , \tag{8}$$

where M is the number of points in the integration segment, and $\lambda^{-1} \sim N^{s/(4-s)}$ is the size scale of an asymptotic soliton. This fairly stringent condition imposed on the parameters of the numerical model and initial state (4) was always satisfied in our calculations. The calculations were carried out on the ES-1037–ES-2706 multiprocessing complex, ICR, Academy of Sciences of the USSR. Equation (1) was integrated, using the BPF algorithms, in accordance with the procedure similar to that used in

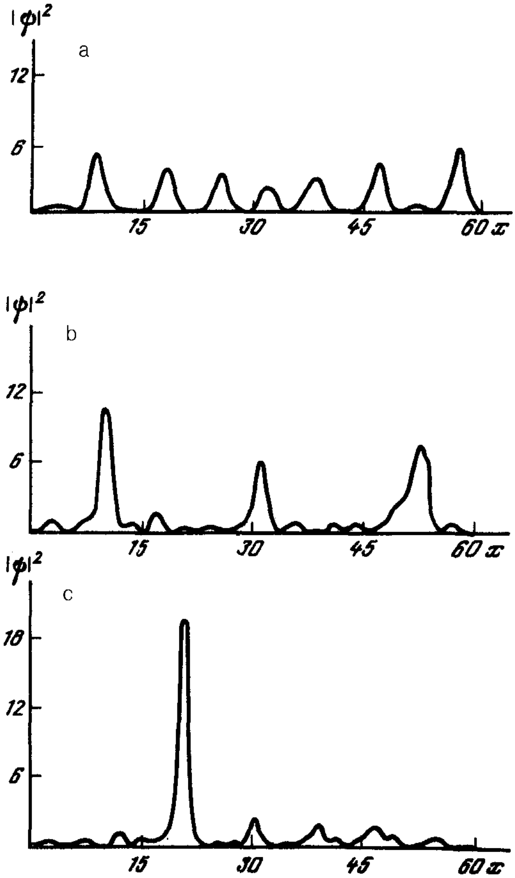

FIG. 1. Fragments of the solution of the equation $i\psi_t + \psi_{xx} + |\psi|\psi = 0$ with the initial-condition parameters $\psi_0 = 1$ and $L = 60$. a—$t = 17.4$; b—$t = 365.4$; c—$t = 730.8$.

Ref. 9. To check the work, we used the integrals of motion in (3). In addition to the power-law nonlinearity, we analyzed systems with a saturation: $f(u) = u(1 - au)$ and $f(u) = u(1 + b_1u)/(1 + b_2u)$.

The results of the calculations showed that the space-time dynamics of the system predicted above is in complete agreement with the qualitative picture of the soliton turbulence. Figure 1 shows results for $f(u) = \sqrt{u}$. The development of modulational instability gives rise to the formation of a soliton grating with a period of order λ_{mod} (Fig. 1a). With further evolution, the system decays progressively more efficiently into solitons and slightly nonlinear free waves. The interaction of solitons with each other and with free waves accounts for the gradual transfer of waves from the weak solitons to stronger solitons. As a result, the amplitude of the solitons increases as their number decreases (Fig. 1b). At large time scales the system reaches a state in which it has a single soliton of small size and large amplitude (Fig. 1c). The measured velocity v of a single soliton is much lower than the group velocity $(\partial\omega/\partial k)_{k=\lambda}$, as should be the case : A soliton at rest generates the lowest energy. An asymptotic soliton is

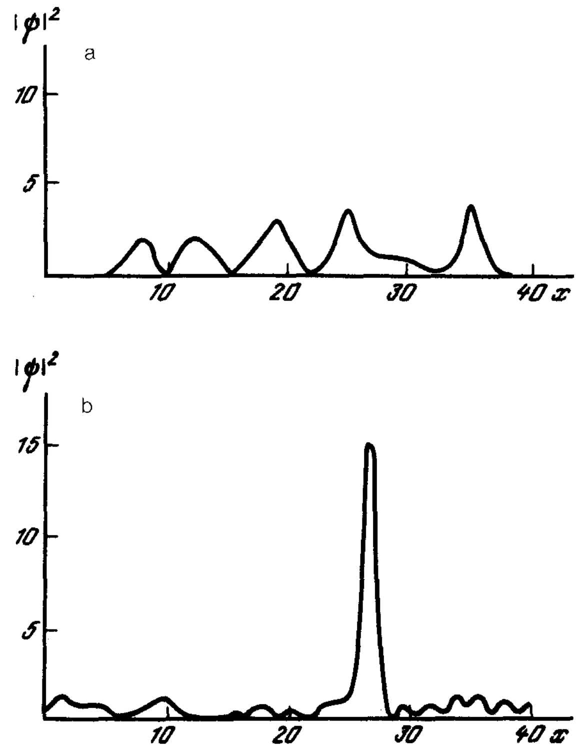

FIG. 2. Fragments of the solution of the equation $i\psi_t + \psi_{xx} + |\psi|^2[(1+0.1|\psi|^2)/(1+0.5|\psi|^2)]\psi = 0$ with the initial-condition parameters $\psi_0 = 1$ and $L = 40$. a—$t = 52.2$; b—$t = 765.4$.

adequately described by an exact solution of (6), in which λ is calculated from the measured amplitude. The free waves account for about 15–20% of the initial integral of N.

Similar results were obtained in the case of other types of nonlinearities. A nonlinearity of the type $f(u) = u(1 + 0.1u/1 + 0.5u)$, for example, is shown in Fig. 2. The qualitative picture of the evolution of the turbulence is in agreement with that described above.

3. In the evolution of the persistent soliton turbulence the nonintegrable system is attracted to the soliton solution, making it possible for the statistic attractor to announce the soliton. In real, extended systems with a dissipation the soliton amplitude increases and the size scale decreases to the point at which small-scale damping begins. This means that the soliton wave turbulence agrees qualitatively with the turbulence in which the energy is transported by collapsing cavities.

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