Berry's phase and the chiral anomaly

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There is a correspondence between the global chiral anomaly and the chiral properties of a fermion Berry's phase.

The discovery of nontrivial adiabatic phases[1,2] has attracted considerable interest since it has opened up a unique opportunity to find a unified description of effects which differ in physical nature (see Ref. 3 for a review), including the guage anomalies in quantum field theory.[4–6] In this letter we show that the global chiral anomaly owes its existence to a fermion Berry's phase.

For definiteness we consider quantum chromodynamics (QCD) with a single massless quark. The Langrangian in Minkowski space is

$$L = -\frac{1}{2} G^2_{\mu\nu} + \bar{q} i\hat{D} q - \bar{\Psi}(i\hat{D} - M)\Psi, \quad D_\mu = \partial_\mu + igA_\mu , \tag{1}$$

where A_μ is the gluon field, $G_{\mu\nu}$ is the stress, q is the quark field, Ψ is the regulator field, and g is a coupling constant. Below we assume the gauge $A_0 = 0$. In our case the role of the "slow" variables with respect to the quark degrees of freedom is played by the static gluon field $\mathbf{A}(\mathbf{x})$. The Hamiltonian of the fermions depends on the time through the time dependence of the external fields, so when periodic boundary conditions $\mathbf{A}(0) = \mathbf{A}(T)$ are imposed, we have a closed contour in the functional space of static fields. A quark determinant was derived in Refs. 5 in the form

$$Z(T)^{T \to \infty} = \exp\left\{ \frac{i}{2} \sum_n \int_0^T dt\, |E_n(t)| - \oint \mathbf{C}(\mathbf{A}) d\mathbf{A} \right\} , \tag{2}$$

where the second term has the meaning of a Berry's phase, and the summation is over the eigenstates of the one-particle quark Hamiltonian H_q with energies E_n. After a regularization, the phase can be written as

$$\oint \mathbf{C}(\mathbf{A}) d\mathbf{A} = \oint d\mathbf{A} < vac\,|\, \frac{\delta}{\delta \mathbf{A}} \,|\, vac >, \tag{3}$$

and the induced quark stress in the field space is (there is a corresponding expression for the regulators)

$$F^{ab}_{ij}(\mathbf{x},\mathbf{y}) = \frac{\delta}{\delta A^a_i(\mathbf{x})} C^b_j(\mathbf{y}) - \frac{\delta}{\delta A^b_j(\mathbf{y})} C^b_i(\mathbf{x})$$
$$= \frac{1}{2} \sum_{n,m} \langle n \,|\, \frac{\delta}{\delta A^a_i(\mathbf{x})} \,|\, m \rangle \langle m \,|\, \frac{\delta}{\delta A^b_j(\mathbf{y})} \,|\, n \rangle \{ \operatorname{sign} E_n - \operatorname{sign} E_m \} , \tag{4}$$

where i, j are Lorentz indices 2nd a, b are color indices, and $|n\rangle$ are the eigenfunctions of H_q. The vacuum state is defined as the direct product of the quark vacuum $|\text{vac}\rangle_q$ and the regulators $|\text{vac}\rangle_\Psi$: $|\text{vac}\rangle := |\text{vac}>_q \otimes |\text{vac}\rangle_\Psi$. The one-particle Hamiltonian of massless quarks has a symmetric spectrum, since the equality $\{H_q,\gamma_0\}_{ab} = 0$, which is equivalent to the condition $H = -H^*$, holds.[1,3] We thus have $F^{ab}_{ijq} = 0$ and $C_q = (\delta\chi/\delta\mathbf{A})$ away from points in the functional space at which H_q is degenerate. In the absence of gauge anomalies, the regulators also have[3,7] $F_\Psi = 0$, and the total Berry connection $\mathbf{C} = \mathbf{C}_q + \mathbf{C}_\Psi$ is a purely gauge connection away from degeneracy points.

To find the chiral anomaly, we use the change of variables $q \to e^{i\alpha(x)\gamma_5} q$, $\Psi \to e^{i\alpha(x)\gamma_5}\Psi$ in the functional integral; from the condition that the regularized determinant $Z^{\text{reg}}(T)$ not change we then find the equation $\partial_\mu j^{\text{reg}}_\mu = -2M\bar{\Psi}\gamma_5\Psi$. The right side can evidently be generated by a global transformation with a constant α, so a manifestation of the existence of an anomaly would be an invariance of Berry's phase under a global chiral rotation. Under a global rotation, the state vectors in the Hilbert space transform as $|\text{vac}\rangle_q \to e^{-i\alpha Q_{5q}} |\text{vac}\rangle_q$; $|\text{vac}\rangle_\Psi \to e^{-i\alpha Q_{5\Psi}} |\text{vac}\rangle_\Psi$, where Q_{5q} and $Q_{5\Psi}$ are the operators of the conserved chiral charges. The charge Q_{5q} is not a functional of the external field, in contrast with $Q_{5\Psi}$, which contains an anomalous term which is related to the mass of the regulator:

$$Q_{5\Psi} = \int d^3x \Psi^+ \gamma_5 \Psi + Q^{\text{an}}_{5\Psi} , \tag{5}$$

where the additional term is of the standard form,

$$Q^{\text{an}}_{5\Psi} = -2\omega_0(\mathbf{A}) = \frac{g^2}{8\pi^2} \int d^3x \epsilon^{ijk} \operatorname{tr}[G_{ij} A_k - \frac{2}{3} A_i A_j A_k] . \tag{6}$$

As a result, under a global chiral rotation Berry's phase changes by an amount

$$i\alpha \oint d\mathbf{A} \frac{\delta Q_{5\Psi}}{\delta \mathbf{A}} = i\alpha \int dt dx \frac{d\mathbf{A}(\mathbf{x})}{dt} \frac{\delta Q_{5\Psi}}{\delta \mathbf{A}(\mathbf{x})} , \tag{7}$$

in agreement with the expression for the anomaly. We wish to stress that, in contrast with the situation regarding a gauge local anomaly, where we have $F \neq 0$, in our example we have run into a Bohm-Aharanov effect in field space.

If we assume that the θ term is contained in the QCD Lagrangian,

$$L' = L + \frac{\theta g^2}{16\pi^2} G_{\mu\nu} G^*_{\mu\nu} , \quad G^*_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G_{\alpha\beta} , \tag{8}$$

we find that the canonical momentum of the gauge field is $\vec{\pi} = (dA/dt) + (\theta g^2 \mathbf{B}/8\pi^2)$, where $\mathbf{B}$ is the magnetic field. The presence of Berry's phase alters the canonical momentum in the following way: $\vec{\pi} \to \vec{\pi}' = \vec{\pi} - i\langle \text{vac}|\delta/\delta A|\text{vac}\rangle$. The Schrödinger equation for the gauge-invariant wave function of the ground state in a second-quantized theory, $\Phi_0(\mathbf{A}) = \varphi(\mathbf{A})|\text{vac}\mathbf{A}\rangle$, is

$$H\Phi_0(\mathbf{A}) = \epsilon_{vac} \Phi_0(\mathbf{A}) , \tag{9}$$

and in the Born-Oppenheimer approximation the Schrödinger equation for $\varphi(\mathbf{A})$ contains $H_{\rm eff} = \int d^3x(\pi'^2/2 + V)$, where V is a potential-energy term. In other words, the substitution $\vec{\pi} \to \vec{\pi}'$ can be pursued if we recall that the fermions generate a term $\int d^4x(dA/dt)C(\mathbf{A})$ in the effective Lagrangian of the gluon field. In $H_{\rm eff}$ we make the substitution $|{\rm vac}\rangle_q \to e^{i\theta\omega_0(A)}|{\rm vac}\rangle_q$, $|{\rm vac}\rangle_\psi \to e^{i\theta\omega_0(A)}|{\rm vac}\rangle_\psi$; here the transformation properties of the complete vacuum wave function do not change under a global gauge transformation. When we make this substitution, we find that an additional term is induced in $S_{\rm eff}$; this additional term is equal to the change in the Barry's phase: $\Delta\oint(C_q + C_\psi)d\mathrm{A}$. The one-particle Hamiltonian of the quarks has a symmetric spectrum, so the quark phase must be quantized[1,3]: $i\oint C_q d\mathrm{A} = \pi n, n \in Z$. Since θ is an arbitrary number, and the term induced by the quarks is $\theta\oint d\mathrm{A}\delta\omega_0/\delta\mathrm{A} = \theta k, k \in Z$ {if the contour is closed in the orbit space $\mathbf{A}(T) = [\mathbf{A}(0)]^{g_k}$, where g_k is an element of the gauge group with topological charge k}, in the case $\theta \neq \pi$ we find the condition $\Delta\oint\mathbf{C}_q d\mathbf{A} = 0$. The condition $k = 0$ corresponds directly to the vanishing of the fermion determinant of massless quarks in a topologically nontrivial external field.[8] For a Berry's phase of regulator fields, there is no quantization condition, and when we take the relation $\delta\omega_0/\delta\mathbf{A} = (1/8\pi^2)\mathbf{B}$ into account, we find that the term which it induces cancels out the initial (seed) θ term.

In summary, a Hamiltonian interpretation of the independence of $\epsilon_{\rm vac}$ from θ in a theory with a massless quark has been proposed.

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