Calculation of a scalar determinant in the theory of open strings

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A direct calculation of the determinant of a scalar Laplacian on a Riemann surface with an edge in terms of the determinant of a scalar Laplacian on a double and the matrix of periods of the double is proposed.

1. In the first-quantization formalism for open strings, integrals over fields on Riemann surfaces with an edge are evaluated. A Riemann surface Σ with p handles and an edge Γ consisting of $m+1$ components can be represented as a factor space of a closed Riemann surface D of type $2p+m$ (called a "double") in terms of an antiholomorphic Z_2 isometry: $\Sigma = D/Z_2$. The boundary Γ consists of the fixed points of this isometry; the Z_2 symmetry of the double induces a Z_2 symmetry of the holomorphic differentials on it: With the differential $w(z)$ one associates a differential $w^*(z) = \sqrt{w(z^*)}$, where z^* is the transform of z in the case of Z_2 symmetry.

The space of all functions on the double is equal to the direct sum of the spaces of Z_2-even and Z_2-odd functions, so a determinant on a double is equal to the product of determinants calculated from even and odd functions. The functions on Σ with a zero normal derivative at the boundary (which satisfy the boundary condition for open strings) are continued in a single-valued way to even functions on a double, while functions on Σ which vanish at the boundary are continued to odd functions, so we

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have

$$\det{}'_D \Delta = \det{}'_{open} \Delta \cdot \det_0 \Delta . \tag{1}$$

On the other hand, if we write $\det'_D\Delta$ as a path integral over Grassmann fields Φ and evaluate it, integrating first over the fields Φ with fixed values Φ_Γ on Γ and then over all such values, we find

$$N^{-1} \det{}'_D \Delta = K \det{}_0^2 \Delta ,$$

$$K = n^{-1} \int' D\bar{\Phi}_\Gamma D\Phi_\Gamma \exp S_K, \quad S_K = i \int (\partial\bar{\Phi} \wedge \bar{\partial}\Phi - \bar{\partial}\bar{\Phi} \wedge \partial\Phi), \tag{2}$$

where $\Phi_h = \Phi_h(\Phi_\Gamma)$ is a harmonic function on Σ, equal to Φ_Γ on Γ. The double area N and the boundary length n constitute the normalization of the zero modes.

It follows from (1) and (2) that a scalar determinant in the theory of open orientable strings can be expressed in terms of a scalar determinant on a double in the following way:

$$N^{-1} \det{}'_{open} \Delta = (KN^{-1} \det{}'_D \Delta)^{1/2} . \tag{3}$$

We calculate K explicitly in this letter (other methods for doing so are described in Refs. 1–3).

2. An arbitrary harmonic function on Σ can be written unambiguously in the form

$$\Phi_h = f(z) + g(\bar{z}) + \sum_{k=0}^{2p+m} c_k F_k, \quad F_0 = 1, \quad F_k = \int^z (w_k + \sum_r \alpha_{kr} \bar{w}_r), \; k = \overline{1, 2p+m}, \tag{4}$$

where $f(z)$ is a holomorphic function, $g(\bar{z})$ is an antiholomorphic function (neither is a constant), w_k are canonical differentials on the double, and the coefficients α_{kr} are chosen in such a way that the function F_k is single-valued on Σ.

In this section of the letter we propose a method for calculating K, in which it is obvious that the contribution to K from time $f(z)$ and $g(\bar{z})$ is equal to a constant which does not depend on the moduli.

We switch to an integration over all the harmonic functions in (2), replacing S_K by $S_K + R$, where R is a zero-mode regulator which does not depend on the metric.

The determinant of an operator which differentiates along the boundary is like K in that it can be written as a path integral over all of the harmonic functions:

$$I = \int D\bar{\Phi}_\Gamma D\Phi_\Gamma \exp(S_I + R), \quad S_I = i \int_\Gamma \bar{\Phi}_h (\partial + \bar{\partial}) \Phi_h . \tag{5}$$

Since I depends on neither the metric nor the moduli of surface Σ, the calculation of K reduces to a calculation of the ratio K/I. Transforming in the numerator and denominator of the ratio K/I from the measure $|D\Phi_\Gamma|^2$ to the measure $D\Phi_h|^2 = |DfDg\Pi_{k=0}^{2p+m} dc_k|^2$, which corresponds to representation (4) [here $|Df|^2$ ($|Dg|^2$) is some measure on the space of (anti-) holomorphic functions], we find

$$\frac{K}{I} = \frac{\int |D\Phi_h|^2 \exp(\tilde{S}_K - \int i\bar{f}\partial f + \int i\bar{g}\bar{\partial} g)}{\int |D\Phi_h|^2 \exp(\tilde{S}_I + \int i\bar{f}\partial f + \int i\bar{g}\bar{\partial} g)}, \tag{6}$$

where $\tilde{S}_M = \Sigma \bar{c}_k c_e S_M(\bar{F}_k, F_e) + R, M = I, K$. In deriving (6) we used the representation of S_K as an integral over the boundary $i\int_\Gamma \sim \Phi_h(\bar{\partial} - \partial)\Phi_h$, and also the circumstance that the integrals of $\bar{g}\partial f$, fw, and $f\bar{w}$ over the boundary are zero since they are integrals of differentials which are holomorphic on Σ (under the restriction $\bar{w} = w^*$ at the boundary).

An integration over holomorphic and antiholomorphic functions in (6) yields a unit contribution (without a sign) to the ratio K/I.

3. To complete the calculation of K, we need to find $\tilde{S}_k$ and $\tilde{S}_I$. This is essentially a matter of finding the matrix $\int_\Sigma \bar{w}_i \Lambda w_j$. We will solve this problem in this section of the letter by generalizing the Riemann relations to the case of a surface with an edge.

On a double we choose a system of cycles $A_i, B_i, i = \sqrt{1, 2p + m}$ of such a nature that for $i = \sqrt{1, p}$ the quantity A_i lies at Σ; for $i = p + 1, p + m$ it is one of the components of the boundary; and the cycle A_{p+m+i} is Z_2-symmetric with respect to the cycle A_i. The matrix of periods of the double is then

$$T_{ij} = i \begin{vmatrix} a & b & c \\ b^T & t & \bar{b}^T \\ \bar{c} & \bar{b} & \bar{a} \end{vmatrix}, \tag{7}$$

where t is a real, symmetric $m \times m$ matrix, and c is a Hermitian $p \times p$ matrix. cutting Σ into cycles $\gamma = (A_i, B_i, \frac{1}{2} B_k)$, $\sqrt{i = 1}, p$; $k = \sqrt{p + 1, p + m}$ ($B_k/2$ is the half of cycle B_k which lies in Σ), we write the differential u, which is holomorphic on the double, in the form $u = \partial f_u$. Carrying out transformations similar to those of Ref. 4, and noting that for the differential w, which is holomorphic on the double under the restriction $\bar{w} = w^*$ on Γ, we find

$$\int_\Sigma \bar{w} \wedge u = \sum_\gamma \int_\gamma (\bar{w} - w^*) \int_{\tilde{\gamma}(\gamma)} u, \tag{8}$$

where for $\gamma = (A_i, B_i, \frac{1}{2} B_k)$ we have $\tilde{\gamma}(\gamma) = (B_i, -A_i, -A_k)$. From (7) and (8) we find

$$\int_\Sigma \bar{w}_i \wedge w_j = i \begin{vmatrix} 2\,\mathrm{Re}\, a + \bar{c} & 2\,\mathrm{Re}\, b & c \\ 2\,\mathrm{Re}\, b^T & t & 0 \\ c & 0 & -c \end{vmatrix}. \tag{9}$$

Relations (9) generalize the Riemann relations to the case of a surface with an edge.

Substituting (9) into (6), we find the final expression for K

$$K = I \det(\mathrm{Re}(a-c)) / \det\begin{pmatrix} t & 2\,\mathrm{Re}\,b^T \\ \mathrm{Re}\,b & \mathrm{Re}(a+c) \end{pmatrix}. \tag{10}$$

We note in conclusion that surfaces with an edge arise in multiloop string calculations based on the cutting of a surface of high type into simpler parts. The technique proposed in this letter may prove useful for that approach.

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