Conformal blocks of minimal models

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Prototype conformal blocks for minimal models are constructed through a bosonized extension of these models.

1. Some progress has recently been achieved in research on conformal theories on arbitrary closed Riemann surfaces.[1,2] The basic entities in the corresponding calculus are "conformal blocks," i.e., holomorphic cross sections of certain stratifications on spaces of moduli which are associated with the theory under consideration. Certain general properties of these entities are discussed in Refs. 1, but, aside from a few exceptions, we do not yet know any explicit examples of conformal blocks. All of the exceptional cases are in one of the following categories: minimal models (the correlation functions are known for type 0 and are characteristic for type 1; Refs. 3 and 4), b–c systems (the correlation functions are known for an arbitrary type), and certain simple generalizations such as toric and orbifold compactifications. In this letter we suggest the use of a bosonization procedure for minimal models[5] with central charges $c = 1 - 24\alpha_0^2 = 1 - 6(m-n)^2/mn$ for an arbitrary type. The procedure would be used to construct expressions which can serve as material for constructing conformal blocks in these theories.

2. The dimensionalities of the primar fields in minimal models $M_{n,m}$ (n and m are two mutually simple positive integers) are given by the Kac formula

$$h_{r,s} = \frac{(mr - ns)^2 - (m-n)^2}{4mn} \tag{1}$$

under the conditions $1 \leqslant r \leqslant n-1$, $1 \leqslant s \leqslant m-1$. Without these conditions and some further restrictions on the states of higher levels, which follow from the existence of null vectors,[3] expression (1) is the same as the spectrum of a real boson field ϕ which takes on values on a circle of radius $R = \sqrt{2mn}$. We consider the corresponding string action:

$$S = (2\pi)^{-1} \int d^2 z (\partial\phi\bar{\partial}\phi - {}^1/_2 i\sqrt{2}\alpha_0 \phi\sqrt{g}\mathscr{R}) \tag{2}$$

(g and $\mathscr{R}$ are the metric on the world sheet and its curvature). The values of the fields ϕ and $\phi + 2\pi R$, $R = \sqrt{2mn}$, are assumed to be identical. The energy-momentum tensor is $T_\phi = -\frac{1}{2}(\partial\phi)^2 - i\sqrt{2}\alpha_0\partial^2\phi$. The value of R is determined by the single-valuedness of $\exp(-S)$ when ϕ and $\phi^+ 2\pi R$ are identical: We need to allow for the circumstance that $(4\pi)^{-1}\int\sqrt{g}\mathscr{R} = (2p-1)$ is an integer for closed and open Riemann surfaces. The primary vortex operators are of the form $|\alpha\rangle \sim \exp(i\sqrt{2}\alpha\phi)|0\rangle$, and the condition for single valuedness at $R = \sqrt{2mn}$ allows α to take on values $\alpha_{r,s}$

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$= \frac{1}{2}(r-1)\alpha_+ + \frac{1}{2}(s-1)\alpha_-$ with $\alpha_+ = \sqrt{m/n}$, $\alpha_- = -\sqrt{n/m}$ (i.e., $\alpha_\pm = \alpha_0 \pm \sqrt{1+\alpha_0^2}$) and with any integers r and s. The dimensionality $\alpha^2 - 2\alpha\alpha_0$ of these operators with respect to T_ϕ is given by Kac formula (1).

It is a straightforward matter to calculate the functional integral

$$\int D\phi e^{-S} \prod_i e^{i\sqrt{2}\alpha_i \phi(\xi_i)} \sim \left[\frac{\exp^{1/24 c S_L}}{(\det \Delta_0)^{1/2}} \sum_{\vec{\delta}} |F_{\vec{\delta}}|^2\right] \delta\left(\Sigma \alpha_i + 2\alpha_0(p-1)\right) \qquad (3)$$

(the calculation is no different from that given in Refs. 6 and 7). Here S_L is the Liouville action, p is the type of world surface, and

$$F_{\vec{\delta}}\{\xi\} \sim \prod_{i<j} E(\xi_i \xi_j)^{2\alpha_i \alpha_j} \sigma_*(\xi_i)^{-4\alpha_i \alpha_0}$$
$$\times \theta\left[\begin{matrix}\delta \\ 0\end{matrix}\right] (2\sqrt{mn}(\Sigma \alpha_i \vec{\xi}_i + 2\alpha_0 \vec{\Delta}) | 2mnT) F_* . \qquad (4)$$

This relation is literally correct when the metric on the world surface is $g = |\nu_*|^4$, i.e., when it is singular and has double zeros at the points $R_1^*,...,R_{p-1}^*$. The quantity $E(\xi_i,\xi_j)$ in (4) is a Prime bidifferential; $\sigma_*(\xi) \equiv \nu_*(\xi)/\Pi_\alpha E(\xi, R_\alpha^*)$; $F_* \equiv \Pi_{\alpha<\beta} E(R_\alpha^*, R_\beta^*)/\Pi_\alpha \nu_*'(R_\alpha^*)$; the θ-function in (4) is of order $2mn$; and the theta characteristic $\vec{\delta}$ takes on values in the set $[Z_{2mn} \equiv (2mn)^{-1} Z (\bmod 2mn)]^{\otimes p}$. The expressions $F_{\vec{\delta}}$ are the prototype conformal blocks which we are seeking. We note that the expressions $F_{\vec{\delta}}\{\xi\}$ change by phase factors upon displacements of ξ by A_M periods, while during a displacement of ξ_i by a period B_M the expression $F_{\vec{\delta}}$ converts into another conformal block:

$$F[\delta_1 \ldots \delta_p] \to F[\delta_1 \ldots \delta_M + \frac{(r_i-1)m - (s_i-1)n}{2mn} \ldots \delta_p] . \qquad (5)$$

3. Bosonized theory (2) is of course more than a minimal model. To construct a minimal model, we would have to impose yet more restrictions. Their meaning, however, is only to distinguish an **irreducible** representation of a Virasoro algebra. This is a linear algebra, so the correct conformal blocks are linear combinations of $F_{\vec{\delta}}\{\xi\}$. Actually, this operation is not completely trivial since in those cases in which it is possible to integrate over some ξ_i, i.e., in which the corresponding dimensionality is unity, the integration should also be thought of as one of the linear operations required in order to distinguish an irreducible representation. The condition $\Sigma_i \alpha_i + 2\alpha_0(p-1) = 0$, which arises from the integration over a constant mode in (3), means that there are zero modes—analogs of holomorphic differentials—in the chiral version of the bosonized theory. From this standpoint, in order to eliminate the zero modes it would be natural to include a suitable number of operators of dimensionality 1, $\exp i\sqrt{2}\alpha_\pm \phi$, among the primary operators on the left side of (3). Integrating over the positions of these operators along contours which cannot be contracted, we find the next approximation to the conformal blocks of the theory. The actual conformal blocks are discrete linear combinations of these entities. They satisfy the equations[2] which follow from the

conditions for the existence of zero vectors. Let us take a brief look at the derivation of these equations for an arbitrary type.

4. Let us consider the correlation functions of the energy-momentum tensors of the minimal model, T, and of its bosonized version (2), T_ϕ:

$$G(\xi) \equiv \langle T(\xi)\rangle; \qquad G_\phi(\xi) \equiv \langle T_\phi(\xi)\rangle; \qquad G(\xi_1,\xi_2) \equiv \langle T(\xi_1)T(\xi_2)\rangle; \qquad G_\phi(\xi_1,\xi_2) \equiv \langle T_\phi(\xi_1)T_\phi(\xi_2)\rangle. \tag{6}$$

The difference between the two expectation values from the first line here is a holomorphic quadratic differential: The gravitational anomaly cancels out. It is therefore a linear combination of 2-differentials $f_1(\xi) \,..\, f_{33p-3}3(\xi)$, which are related to the moduli y_k of the surface under consideration: $g^{(1)}(\xi) \equiv G(\xi) - G_\phi(\xi) = \Sigma_{k=1}^{3p-3} a_k f_k(\xi)$. The coefficients a_k can be expressed in terms of the derivatives of the character (χ) of this theory

$$\frac{\partial \ln \chi}{\partial y_k} = \int \langle T(\xi)\rangle \eta_k(\xi) \Rightarrow a_k = \sum_{l=1}^{3p-3} a_l \int f_l \eta_k = \frac{\partial \ln \chi}{\partial y_k} - \frac{\partial \ln \chi_\phi}{\partial y_k}.$$

For further applications to higher-order correlation functions, we note that nearly everywhere in the modulus space the holomorphic quadratic differentials are bilinear combinations of linear differentials: $f_k(\xi) = \Sigma_{\mu,\nu=1}^{p} F_{k\mu\nu}\omega_\mu(\xi)\omega_\nu(\xi)$. Accordingly, we can write

$$g^{(1)}(\xi) = \sum_{k=1}^{3p-3} a_k f_k(\xi) = \sum_{\mu,\nu=1}^{p} A_{\mu\nu}\,\omega_\mu(\xi)\,\omega_\nu(\xi); \quad A_{\mu\nu} = \sum_{k=1}^{3p-3} a_k F_{k\mu\nu}.$$

This representation makes it possible to determine the symmetric holomorphic (1,1) bidifferential $g^{(1)}(\xi_1,\xi_2) \equiv \Sigma_{\mu,\nu=1}^{p} A_{\mu\nu}\omega_\mu(\xi_1)\omega_\nu(\xi_2)$, which will be useful below.

In the case of a two-point correlation function, we can consider the following anomaly-free combination:

$$g^{(2)}(\xi_1,\xi_2) \equiv G(\xi_1,\xi_2) - G(\xi_1)G(\xi_2) - G_\phi(\xi_1,\xi_2) + G_\phi(\xi_1)G_\phi(\xi_2).$$

This meromorphic quadratic bidifferential has poles on the diagonal. Its singular part is dictated by the operator expansion

$$T(\xi_1)T(\xi_2) \sim \frac{c/2}{(\xi_1-\xi_2)^4} + \frac{2T(^1/_2(\xi_1+\xi_2))}{(\xi_1-\xi_2)^2} + 0(1).$$

It follows that we have

$$g^{(2)}(\xi_1,\xi_2) \sim \frac{2g^{(1)}(^1/_2(\xi_1+\xi_2))}{(\xi_1-\xi_2)^2} + 0(1) \sim \frac{2\tilde{g}^{(1)}(\xi_1,\xi_2)}{(\xi_1-\xi_2)^2} + 0(1),$$

so the (2,2)-bidifferential is $g^{(2)}(\xi_1,\xi_2) = 2\omega(\xi_1,\xi_2)\tilde{g}^{(1)}(\xi_1,\xi_2)$

$+\Sigma_{k,l=1}^{3p-3} a_{kl} f_k(\xi_1) f_l(\xi_2)$. Here $\omega(\xi_1,\xi_2) \equiv \partial^2 \ln E(\xi_1,\xi_2)/\partial\xi_1\partial\xi_2$ represents a standard symmetric (1,1)-bidifferential with a second-order pole on the diagonal. The coefficients a_{kl} can be expressed in terms of the derivatives of the character: $\partial^2 \ln\chi/\partial y_k \partial y_l$ and $\partial \ln\chi/\partial y_k \times \partial \ln\chi/\partial y_l$.

It is thus possible to determine all the multipoint correlation functions T in terms of χ by a recurrence procedure.

The minimal model is specified by the specific condition for the existence of a "zero vector," which can be written as an equation for the expectation value of the energy-momentum tensor and its derivatives at coincident points:

$$\langle P(T) \rangle (\xi) = 0 \tag{7}$$

[for example, for $M_{2,5}$ we would have $P(T) = \partial^2 T - (10/3) T^2$; Ref. 2]. After the expressions found for the correlation functions T in Ref. 7 are substituted in, we find a homogeneous equation for the character χ. Equation (7), which is a holomorphic differential in ξ, can be expanded in a basis of corresponding differentials. The result is a finite set of equations which was proposed by Zamolodchikov.[2] The various solutions of these equations are the various characters of the theory. In a corresponding way, one can derive equations for arbitrary conformal blocks. Knowledge of a bosonized theory is useful but not necessary for such a derivation. The true usefulness of the bosonized theory is that boson blocks (4) form a narrow class of cross sections whose linear combinations (including integral combinations) include the solutions of the Zamolodchikov equations which are not yet known (for $p \geqslant 2$). (The known expressions for single-loop characters[4] of course satisfy these equations.)

It would be very interesting to also take the next step in another direction: to bosonize any conformal theories in the same "excessive" fashion. Here it would be useful to consider more-complex compactifications (e.g., of the orbifold type). In a sense, it would then become possible to find a unified description of conformal theories without the need to classify irreducible representations of a Virasoro algebra.

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