Chirality transformation and structure of momentum space

A. S. Gorskiĭ
Institute of Theoretical and Experimental Physics

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A chirality transformation for massless one-particle states can be defined in a consistent way for an arbitrary spin, as a translation along a closed contour in momentum space.

The observation of a nontrivial topological phase in the adiabatic approximation,[1,2] which usually originates from a crossing of energy levels at certain parameter values, has stimulated research on the structure of a spectrum as a function of the parameters in various physical problems. One of the clearest examples of a manifestation of a Berry topological phase is optical interference in a system of two helical waveguides,[3,4] in which the role of the parameter is played by the wave vector of a photon, which traces out a closed contour in momentum space as the wave propagates. It was demonstrated in Ref. 5 through an analysis of the structure of the Poincaré group that the momentum-space motion of a massless neutral particle with a spin occurs in the effective field of a monopole which has a unit magnetic charge and which is at the point $\mathbf{p} = 0$. We will show below that the presence of a monopole has the consequence that the translation of the wave function of a particle of arbitrary spin along a closed contour $|\mathbf{p}| = \text{const}$ in momentum space is equivalent to a chirality transformation.

We consider a free, massless, spin-1/2 particle, whose dynamics is described by the Dirac equation

$$-\frac{\partial \Psi}{\partial t} = (\vec{\alpha}\vec{\nabla})\Psi, \qquad \alpha = \vec{\gamma_0}\vec{\gamma}, \tag{1}$$

where $\Psi(\mathbf{x},t)$ is a four-component wave function. Equation (1) has the one-particle plane-wave solution $\Psi_{\mathbf{p}}(\mathbf{x},t) = u_{\mathbf{p}}(t)e^{i\mathbf{p}x}$, where $u_{\mathbf{p}}$ is the standard bispinor. As four independent solutions with a fixed momentum we choose solutions characterized by definite values of the energy $E = \pm|\mathbf{p}|$ and of the chirality $\lambda = \pm 1/2$:

$$u_{\underset{E\ \lambda}{++}} = \begin{pmatrix} u_{++} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u_{+-} = \begin{pmatrix} 0 \\ 0 \\ u_{+-} \\ 0 \end{pmatrix},$$

$$u_{-+} = \begin{pmatrix} 0 \\ u_{-+} \\ 0 \\ 0 \end{pmatrix}, \qquad u_{--} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_{--} \end{pmatrix},$$

where we have used the spinor representation of the Dirac matrices. We are thus dealing with a two-level system; each level is doubly degenerate in terms of chirality. There is a crossing of levels at the point $\mathbf{p}=0$; this event gives rise to a monopole at this point, according to Ref. 5. As a closed contour is traced out in momentum space, the solutions of the Schrödinger equation acquire a nondynamic phase, according to Berry. This phase depends on the sign of the energy and the chirality:

$$u_{+\lambda} \to u_{+\lambda} e^{i\lambda\Omega(\mathbf{p})}, \qquad u_{-\lambda} \to u_{-\lambda} e^{-i\lambda\Omega(\mathbf{p})}, \tag{2}$$

where $\Omega(\mathbf{p})$ is the solid angle which is traced out as the vector $\mathbf{p}$ is rotated. Expressed in a different way, this point can be understood by recalling that the operator which performs a pure rotation in momentum space is of the form

$$\mathbf{M} = [-i\mathbf{p}, \frac{\partial}{\partial \mathbf{p}} + i\lambda \mathbf{A}_M] + \frac{\lambda \mathbf{p}}{|\mathbf{p}|},$$

where $\mathbf{A}_M$ is the potential of the field of a monopole, and λ plays the role of an electric charge in momentum space.[5] We also note that a monopole will induce no transitions in a degenerate level upon a change in chirality.[6] Expanding the wave function in linearly independent solutions (2), we easily see that a translation of the wave function along a closed contour with a small value of $\Omega(\mathbf{p})$ is of the form of a small chiral transformation:

$$\delta \Psi_{\mathbf{p}} = i\frac{\Omega(\mathbf{p})}{2}\gamma_5 \Psi_{\mathbf{p}} \tag{3}$$

As another example we consider a spin-1 particle. Maxwell's equations can be writen in the form

$$\frac{d}{dt}\begin{pmatrix} \mathbf{E}+i\mathbf{B} \\ \mathbf{E}-i\mathbf{B}\end{pmatrix} = \begin{pmatrix} s\vec{\nabla} & 0 \\ 0 & -s\vec{\nabla}\end{pmatrix}\begin{pmatrix} \mathbf{E}+i\mathbf{B} \\ \mathbf{E}-i\mathbf{B}\end{pmatrix}, \tag{4}$$

where the operator $(s_e)_{ij} = -i\epsilon_{ejk}$ represents the spin of the photon. Equation (4) has the form of Schrödinger equation, with conserved momentum. It is not difficult to verify that the combinations $\mathbf{E}+i\mathbf{B}$ and $\mathbf{E}-i\mathbf{B}$ have fixed and opposite helicities. Consequently, by analogy with the $s=1/2$ case, during translation along a closed contour in $\mathbf{p}$ space with a small Ω we have

$$\begin{aligned} \delta(\mathbf{E}_{\mathbf{p}} - i\mathbf{B}_{\mathbf{p}}) &= i\Omega(\mathbf{p})(\mathbf{E}_{\mathbf{p}} - i\mathbf{B}_{\mathbf{p}}) \\ \delta(\mathbf{E}_{\mathbf{p}} + i\mathbf{B}_{\mathbf{p}}) &= -i\Omega(\mathbf{p})(\mathbf{E}_{\mathbf{p}} + i\mathbf{B}_{\mathbf{p}}). \end{aligned} \tag{5}$$

This result is equivalent to the standard dual transformation which is ordinarily taken as the chirality transformation for the photon field. A corresponding analysis can be carried out for higher spins.

The picture drawn here can be generalized to the case of second-quantized fields. For example, for a massless fermion field a plane-wave expansion takes the form $\Psi(x) = \Sigma_{\mathbf{p}} a_{\mathbf{p}} \Psi_{\mathbf{p}}(x)$, where $a_{\mathbf{p}}$ are operators, and the summation is over plane waves with positive and negative frequencies. Within the sum, we carry out a momentum transformation in which each vector **p** is rotated through a closed contour around an axis. All of the vectors trace out the same solid angle. As a result, we have $\delta\Psi(x) = (i\Omega(x)/2)\gamma_5\Psi(x)$ if we assume Ω to be small and to depend on the spatial coordinate. Chiral transformations in a second-quantized theory can thus be identified with local rotations in momentum space.

For massive particles there is no level crossing at real values of the momentum, so a topological phase does not arise in this case. Accordingly, there are no nonremovable topological singularities in the phase space of free massive particles. In a theory with an interaction, the position of a possible point of a level crossing in **p** space is fixed by the particular form of the external field; e.g., positive and negative levels of a fermion in a magnetic field **B** cross in the $\mathbf{p}\perp\mathbf{B}$ plane. A relationship between a chiral current and the structure of the phase space was pointed out in Ref. 7 in a discussion of an effect which occurs in He^3 and which is analogous to a chiral fermion anomaly. A common definition of chirality for various spins may also prove useful in the interpretation of the boson anomalies which have recently been discussed.

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