Semiclassical localization in a magnetic field

G. V. Mil'nikov and I. M. Sokolov
P. N. Lebedev Physical Institute, Academy of Science of the USSR

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The localization of the Landau levels of a particle situated in a smooth random 2D potential is considered. The dependence of the inverse localization length $\gamma(E)$ on the energy and the correlation properties of the potential has been determined on the basis of a semiclassical approximation which corresponds to the strength limit of the magnetic field H.

The localization of the eletronic states in a two-dimensional system in a magnet field is of considerable interest. The quantum Hall effect, for example, requires the presence of both the localized and delocalized states.[1,2] The behavior of the inverse localization length $\gamma(E)$ in such systems was studied in Refs. 3–6. Although $\gamma(E)$

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does not directly determine the transport properties of the system, this quantity must be calculated if only because it is the only characteristic of the system which can be determined very accurately in numerical simulations. We will discuss below the behavior of this quantity in a strong field ($H \to \infty$) or in a smooth potential.[7,8]

The Hamiltonian of the system is

$$\mathcal{H} = \frac{(\mathbf{p} - \frac{e}{c}\mathbf{A})^2}{2m} + V(x, y) \equiv \mathcal{H}_0 + V \quad , \tag{1}$$

where $V(x,y)$ is a random potential. We assume that this is a Gaussian potential, $\langle V \rangle = 0$, $\langle V(\rho) V(\mathbf{r} + \rho) \rangle = g(r)$, and that the correlation length of this potential, λ, is the length scale of the problem. In the Landau gauge the eigenfunctions H_0 are

$$\psi_{X,n} = (2^n n! \sqrt{\pi} l L)^{-1/2} \exp\left(iy \frac{X}{l^2} - \frac{(x - X)^2}{2l^2}\right) H_n\left(\frac{x - X}{l}\right),$$

where $l^2 = \hbar c/eH$, n is the number of the Landau level, X takes on the value $(2\pi l^2/L)\, k$, $(k = 1,2,...)$, and L is the length of the system in the y direction. At low temperatures and in strong fields ($\hbar\omega_H \equiv \hbar(eH/mc) \gg V$) the random field leads to a mixing of only the states which correspond to a single Landau level. To be specific, we will consider the zeroth level. The wave function of the state with an energy E (which is reckoned from $E_0 = \hbar\omega_H/2$) is $\psi = \Sigma_X C_X \Psi_X,0$; the amplitudes C_X satisfy the equation

$$EC_X = \sum_{X'} C_{X'} \langle X | V | X' \rangle , \tag{2}$$

where $\langle X | V | X' \rangle = \int \psi^*_{X,0} V(x,y) \psi_{X',0} dxdy$. As $H \to \infty$, $1 \ll \lambda$, L. In the large system the diagonal matrix element $\langle X | V | X \rangle$ is $\sim L^{-1}$ and tends to zero as $L \to \infty$. In the limit $H \to \infty$ we can replace Eq. (2) by an integral equation

$$EC(x) = \frac{L}{2\pi l^2} \int V(x, \xi) C(\xi) d\xi \quad , \tag{3}$$

where

$$V(x, \xi) = \frac{1}{\sqrt{\pi} lL} \int \exp\left(-i\frac{y}{l^2}(x - \xi) - \frac{(\eta - x)^2 + (\eta - \xi)^2}{2l^2}\right) V(\eta, y) d\eta dy$$

$$\approx \exp\left(-\frac{(x - \xi)^2}{4l^2}\right) V\left(\frac{x + \xi}{2}, \widetilde{\frac{x - \xi}{l^2}}\right). \tag{4}$$

(The wavy bar above the second argument means that the Fourier transform is evaluated over this argument.) Substituting (4) in (3) and seeking a solution in the form $C(x) \sim \exp(i\int^x k(x')dx')$, we find

$$E = V(x, k(x) l^2). \tag{5}$$

Equation (5), which is in agreement with the result obtained in Ref. 9, defines the dispersion relation $k(E,x)$, i.e., it defines the semiclassical behavior of the system, which is applicable if l and k^{-1} are small in comparison with λ. This dispersion relation completely determines the localization in a strong field. The localization length of this state is

$$\gamma(E) = \lim_{|x-x'|\to\infty} \frac{\langle \ln G(x,x',E)\rangle}{|x-x'|}. \tag{6}$$

For the Green's function we use a mixed representation[10] $G(x,x'E) = 1/2\pi\int G(R,k,E)\exp(ikr)dk$, $r=x-x'$, $R=(x+x')/2$. If $k\lambda \gg 1$, $G(R,k,E)$ is given by dispersion relation (5)

$$G(R,k,E) = (E - E(k,R) + i\epsilon)^{-1}, \tag{7}$$

where $E(k,R) = V(R,kl^2)$. Switching to x representation and substituting the result in (6), we find

$$\gamma(E) = \langle \min_n |\operatorname{Im} k_n(E)|\rangle, \tag{8}$$

where $\{k_n(E)\}$ are the roots of dispersion relation (5).

Before calculating the values of γ for a two-dimensional problem, let us consider a typical problem of localization in a smooth, one-dimensional potential with a correlation length λ of the eigenfunctions of the Schrödinger equation $E\psi = -\hbar^2/2m\,\psi'' + U(x)\psi$. In this case we have $E(k,R) = \hbar^2k^2/2m + U(R)$

$$\gamma(E) = \langle |\operatorname{Im}\sqrt{\frac{2m}{\hbar^2}(E - U(x))}|\rangle. \tag{9}$$

In a Gaussian potential, with $E \gg \sqrt{g(o)}$, we have $\gamma(E) = \sqrt{(2\pi m/\hbar^2)E}$, consistent with the result obtained by using the phase formalism.[11] Physically, result (9) means that if $k\lambda \gg 1$, the wave function decays only in the classically forbidden regions. In a constrained random potential $\gamma(E) = 0$, when $E > \max U(x)$, which is consistent with the assertion that all states are localized and which implies that $\gamma \to 0$ as $\lambda \to \infty$ [at $E < \max U(x)$, $\gamma \to \text{const}$].

We now return to the two-dimensional problem. Let us consider the value of E which are close to zero. We call the lines of the $V(x,kl^2) = E$ level the trajectories of the roots. From the symmetry of $V(x)$ it follows that $E=0$ is the percolation threshold of the regions $V<E$ and $V>E$. It therefore follows that there is a line of the $V=0$ level which penetrates the entire system. This is the only line on which an absolute minimum $|\operatorname{Im}(k(E))| = 0$, $\gamma = 0$ is realized. At $E=0$ an increase in γ corresponds to a break of this trajectory at the saddle points of the potential $V(x,y)$ (Fig. 1).[1)] With accuracy to the quadratic terms at these points we have

$$V(x,kl^2) = \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(x-x_0)^2 - \frac{l^4}{2}\frac{\partial^2 V}{\partial y^2}\left(k - \frac{y_0}{l^2}\right)^2, \tag{10}$$

where (x_0,y_0) are the coordinates of the saddle point. If E are small, the trajectory

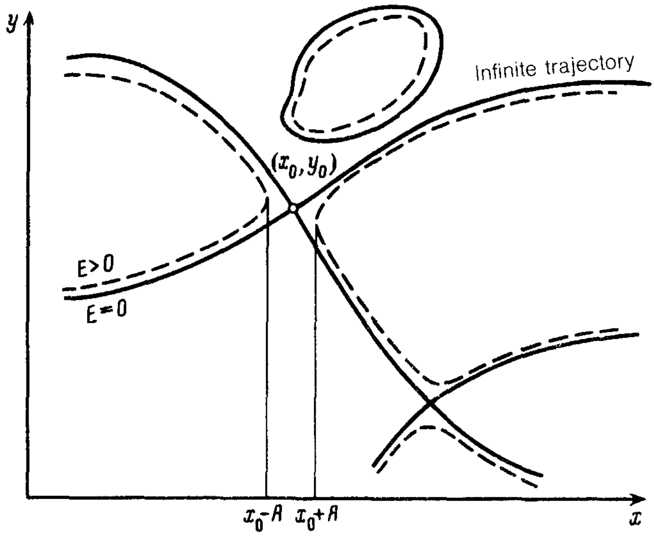

FIG. 1.

splits at these point. The smallest inverse localization length is determined by the appearance of the regions in which $\mathrm{Im}k(x,y) \neq 0$ near these saddle points which correspond to $E \approx 0$. In this case

$$\gamma(E) = \frac{1}{L_{av}} \langle \int_{x_0 - A}^{x_0 + A} \mathrm{Im}\, k(E, x) dx \rangle, \tag{11}$$

where $A = \sqrt{2E/\partial^2 V(x_0,y_0)/\partial x^2}$, and L_{av} is the average difference between the x coordinates of the points at which the trajectory splits.

In terms of the percolation theory, this quantity is a certain characteristic size of a finite cluster near the percolation threshold. This size is proportional to the correlation length $\xi \sim (p - p_c)^{-\nu}$, where in the two-dimensional case we have $\nu = \frac{4}{3}$ (Ref. 10), $p_c = \frac{1}{2}$, and $p = \int_{-\infty}^{E} p(V) dV$. Accordingly, $L_c = B(p - p_c)^{-\nu}$, where the constant B is the functional of the autocorrelation function g. If $g = g(r/\lambda)$, where λ is the correlation length and $B \sim \lambda$, the proportionality coefficient can be determined only numerically.

The integral in (11) can easily be estimated. After the averaging we find

$$\gamma(E) = \frac{E^{\nu+1}\, \Gamma^2(1/4)}{2Bl^2\sqrt{2D_1}\,(2\pi D)^{\nu/2}} \sim \left(\frac{E}{\langle V^2 \rangle^{1/2}}\right)^{\nu+1} \frac{\lambda}{l^2},$$

where $D = g(0) = \langle V^2 \rangle$, and $D_1 = g^{\mathrm{IV}}(0) \sim \langle V^2 \rangle / \lambda^4$. We see that as the potential becomes progressively smoother $\lambda \to \infty$, the correlation length γ^{-1} decreases (in the limit of a static field there is no drift).

In this physical description we have ignored the nonorthogonality of the wave functions along various trajectories, which accounts for the indeterminate energy in dispersion relation (5), $\Delta E \sim \langle V^2 \rangle^{1/2}(l/\lambda)^2$. The result obtained by us is, accordingly, an intermediate asymptotic limit of the dependend $\gamma(E)$ for $\langle V^2 \rangle^{1/2} \gg E \gg \langle V^2 \rangle^{1/2} (1/\lambda)^2$. If h or H is large, the domain of applicability of this asymptotic limit is very broad.

The result obtained by us confirms the conclusions of Ref. 3 that $\gamma(E)$ is a power function, $\gamma \sim |E|^\alpha$. In the limit $\lambda \to \infty$, $\alpha = \nu + 1 = 7/3$, whereas a numerical simulation gives $\alpha \lesssim 2$ for a system with a short-correlated potential.

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[1)] Expressed in a different way, the pattern of the root trajectories reproduces a percolation model which was considered in Ref. 7.

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