Bi-self-similar wave collapse

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A new class of wave-collapse regimes is predicted. This new class makes it possible to reach an understanding of the dynamics of nonlinear wave fields which are prone to self-focusing but which do not have stable self-similar regimes of a strong collapse.

Nonlinear wave fields may, by virtue of a focusing self-effect, increase explosively at certain spatial points to the extent that they reach physically infinite values. This "wave collapse" has been the subject of active research in recent years. Experience has shown that a singularity of a wave field usually forms in a self-similar manner. Depending on how much energy goes into the singular point—a finite amount or an infinitesimal amount—the collapse is called either "strong" or "weak." In the case of a strong collapse, the self-similar solutions make it possible to explain the absorption of energy of the wave field which actually occurs. In a weak collapse, it is not rare to see a paradoxical situation in which energy can be absorbed only at singular points but no significant amount of energy goes to these points. A problem of this type arises, for example, for the nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + |\psi|^2\psi = 0. \tag{1}$$

This equation describes the collapse of the envelope of an intense wave packet in a dispersive medium, serves as a scalar model for the subsonic collapse of plasma waves,

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and has attracted the interest of researchers for a long time now. Equation (1) obviously allows the self-similar substitution

$$\psi = (t_s - t)^{-\frac{1}{2} - i\alpha} \chi(\mathbf{r}/\sqrt{t_s - t}\,). \tag{2}$$

Function (2) has in fact been observed in a numerical calculation (Ref. 1), and later in Ref. 2. A centrally symmetric self-similar solution $\chi(\zeta)$ was found directly in Ref. 3. The establishment of this solution for various initial conditions was tested more carefully in Refs. 4 amd 5. According to those studies, a centrally symmetric self-similar solution exists for $\alpha \approx 0.545$ and has a time-independent asymptotic behavior

$$\psi \approx C/r^{1+2i\alpha} \tag{3}$$

in the region $r \gg \sqrt{t_s - t}$ $(C \approx \sqrt{2})$. In the three-dimensional case (to which we restrict this discussion) a field singularity which arises at the point $r = 0$ as $t \to t_s$ contains a zero energy; i.e., the collapse is weak. Some "semiclassical" self-similar solutions of Eq. (1) have been proposed as candidates for the role of a strong collapse regime,[2,3] but these solutions have proved to be unstable with respect to small-scale perturbations (whose growth is far more rapid than the shrinkage of the basic length scale of the field). No other strong-collapse regimes have been observed so far, although the capture of a finite amount of energy in a self-similarity zone has been observed in a numerical simulation[2] (the rapid formation of a singularity at the point $r = 0$ prevented a study of the subsequent fate of the captured energy).

In the present study we have found a solution for this problem. The underlying idea is extremely simple and appears to be applicable to many cases of weak collapse. Specifically, a singularity which forms during weak collapse does not disappear after the collapse has been completed; instead it continues to exist, and it draws a finite amount of energy to itself. Under conditions such that this process also is of a self-similar nature in the final stage, the dynamics of the wave field might naturally be called "bi-self-similar." The origin of bi-self-similar collapse regimes can also be understood without difficulty from the mathematical standpoint. Specifically, the class of regular self-similar solutions is obviously narrower than the class of self-similar solutions which allow a singularity at a certaim point. The first (weak) collapse lifts the prohibition concerning singular self-similar solutions, and stable self-similar regimes of a strong collapse, initially absent, can be found in a broader class. This is the situation regarding the solutions of Eq. (1). A singular strong collapse which develops after the completion of the weak collapse proceeds in the final stage in accordance with the same self-similar law, (2), as governed the first stage (with, of course, different values of the parameters t_s and α). The function $\chi(\zeta)$ again satisfies the equation

$$\left(-\alpha + \frac{i}{2}\frac{d}{d\zeta}\zeta + \frac{1}{\zeta}\frac{d^2}{d\zeta^2}\zeta + |\chi|^2\right)\chi = 0, \tag{4}$$

but now solutions having a finite energy flux to the singularity are allowed to compete:

$$I = -\lim_{\zeta \to 0} \zeta^2 |\chi|^2 \frac{d}{d\zeta} \arg\chi . \tag{5}$$

Such solutions also exist at $I \gg 1$ and can be found analytically. Of primary interest are solutions with a vanishing value of the coefficient C in asymptotic expression (3) for the wave field at large distances from the singularity. In solutions of this sort, the field amplitude falls off more rapidly in the limit $\zeta \to \infty$ than in regular solutions:

$$|\chi| \underset{\zeta \to \infty}{\to} \text{const}/\zeta^2. \tag{6}$$

The energy integral $\int d\zeta \zeta^2 |\chi|^2$ converges at the upper limit, telling us that all of the energy captured in the self-similar zone is localized in the region $r \lesssim \sqrt{t_s - t}$. By the time t_s this energy has been drawn into the singularity. The singular collapse, in contrast with the preliminary regular collapse, is thus strong.

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