Radiative corrections to the axial anomaly

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The anomaly in the divergence of the axial current is not exhausted by a single loop. A component which has previously been neglected comes from the diagrams representing the scattering of light by light. The radiative corrections are calculated for QED, a non-Abelian gauge theory, and a supersymmetric gauge theory.

For many years it has been believed that the size of the divergence of the axial current is determined by a single loop (the Adler-Bardeen theorem[1]). In supersymmetric theories this divergence appears in one supermultiplet with the trace of the energy-momentum tensor, which is proportional to a β function, so it should not be exhausted by a single loop. Attempts to resolve this paradox now constitute an extensive literature.[2–15]

In the present letter we show that in any theory, including theories which are not supersymmetric, the anomaly in the axial current is of a multiloop nature. Our observation is that the diagrams for the scattering of light by light, which were discarded in Ref. 1 on the basis of dimensionality considerations, cannot be omitted if the mass of the fermion is zero. Actually, the diagram for an anomaly with scattering of light by light (Fig. 1) was estimated in Ref. 1 to be $\sim F\tilde{F}(k_1k_2/m^2)$, where m is the mass of a fermion, and the factor k_1k_2/m^2 stems from the scattering of light by light. If we have $m=0$ (actually, $m^2 \ll |k_1^2|, |k_2^2|, |k_1k_2|$), however, this estimate is incorrect: The amplitude for the scattering of light by light is ~ 1.

We have carried out a direct calculation of the diagrams with a scattering of light by light, of the type in Fig. 1, for QED by the background-field method. Although some complex functions of the external momenta appear in the intermediate expressions, the final result for the amplitude of the transition of two photons, $\partial_\mu j^5_\mu$, has a very simple form:

$$\partial_\mu j^5_\mu = F_{\mu\nu}\tilde{F}^{\mu\nu}\frac{e_0^2}{8\pi^2}\left(1-\frac{3e_0^4}{64\pi^4}\ln\frac{\Lambda^2}{k^2}\right). \quad (1)$$

Here Λ is the ultraviolet cutoff, and the coefficient of the logarithm does not depend on the relation between k_1^2 and k_2^2, and $k^2=(k_1+k_2)^2$. We have omitted some finite terms $\sim e_0^4$ from (1).

The common factor of e_0^2 in (1) transforms into a renormalized charge when radiative corrections to the external photon lines are taken into account. The dependence on the cutoff in parentheses in (1) can be eliminated through a multiplicative renormalization of the operator $\partial_\mu j^5_\mu$. If we multiply both sides of (1) by $Z = 1 + Ce_0^2$ and use $e_0^2 = e^2(k)(1+(1/12\pi^2)\times e^2(k)\ln\Lambda^2/k^2)$, we find that the de-

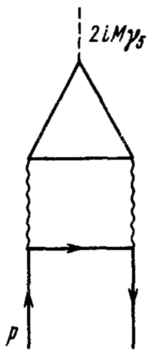

FIG. 1.

pendence on Λ disappears at $C = 9/16\pi^2$. We find

$$(\partial_\mu j^5_\mu)_{ren} = Z(\partial_\mu j^5_\mu) = F_{\mu\nu}\tilde{F}^{\mu\nu}\frac{e^2(k)}{8\pi^2}\left(1 + \frac{9e^2(k)}{16\pi^2}\right). \qquad (2)$$

The need for a renormalization of Eq. (1) is quite natural since the unconserved current j^5_μ appears in this equation.[1)] The renormalization of the divergence $\partial_\mu j^5_\mu$ is determined directly in the lowest nonvanishing approximation by the diagram in Fig. 2, where a regulator fermion field with a mass $M \to \infty$ propagates in the triangle. A direct calculation of the diagram in Fig. 2 yields

$$\partial_\mu j^5_\mu \to \partial_\mu j^5_\mu\left(1 - \frac{3e_0^4}{64\pi^4}\ln\Lambda^2/p^2\right), \qquad (3)$$

where p is the momentum of a massless fermion. Like expression (1), relation (3) requires a multiplicative renormalization with $Z = 1 + 9e_0^2/16\pi^2$. The agreement of the constants Z found from (1) and (3) is a test of the validity of the contribution from the scattering of a photon by a photon which we calculated.

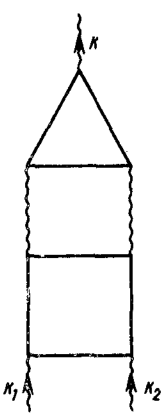

FIG. 2.

What happens if $m \neq 0$? At $m^2 \sim k^2$, in the single-loop approximation, the amplitude of the transition to two photons, $\partial_\mu j^5_\mu$, is[16]

$$\langle 2\gamma | \partial_\mu j^5_\mu | 0 \rangle = \frac{e_0^2}{8\pi^2} (1 + 2m^2 I_{00}) F\tilde{F},$$
$$I_{00}(k_1, k_2) = \int_0^1 dx \int_0^{1-x} dy \, [x(1-x)k_1^2 + y(1-y)k_2^2 + 2xy k_1 k_2 - m^2]^{-1}. \tag{4}$$

A direct calculation of diagrams of the type in Fig. 1 with $m \neq 0$ yields

$$-\frac{3e_0^4}{64\pi^4} \ln \frac{\Lambda^2}{k^2} (1 + 2m^2 I_{00}) \frac{e_0^2}{8\pi^2} F\tilde{F}. \tag{5}$$

As in the massless case, all of contribution (5) is determined by diagrams in which a regulator fermion propagates in a triangle, while a light fermion propagates in a square.

In Eq. (5) we find the same expression $(1 + 2m^2 I_{00})$ as in (4), but, in contrast with (4), it arises in this case from the integration of the square, not of the triangle. It is for this reason that we have a multiplicative renormalizability of the amplitude. The value of Z, of course, does not depend on the relation between m^2 and k^2 here. At $m^2 \gg k^2$ we have $1 + 2m^2 I_{00} \to 0$. In (4), this circumstance reflects the fact that the contributions of the regulator and physical fermions cancel out, while in (5) it corresponds to Adler and Bardeen's arguments regarding the suppression of the diagrams for scattering of light by light by a factor of k^2/m^2.

The equations written above allow a renormalization-group generalization. Using the standard technique (the Kalan-Simanzik equation), we can easily derive the following exact relations:

$$\partial_\mu j^5_\mu = \frac{e^2(k)}{8\pi^2} F\tilde{F} \varphi(e_0^2, \ln \frac{\Lambda^2}{k^2}) = \frac{e^2(k)}{8\pi^2} F\tilde{F} \varphi(e^2(k), 0) \frac{Z(e^2(k))}{Z(e_0^2)}, \tag{6}$$

where

$$Z(e^2) = \exp \int_0^{e^2} \frac{\gamma(e_1^2)}{\beta(e_1^2)} de_1^2, \tag{7}$$

$\beta(e^2)$ is the Gell-Mann–Low function, and $\gamma(e^2)$ is the anomalous dimensionality of the operator j^5_μ.

As we know from Adler and Bardeen's analysis,[1] there is no two-loop correction to the anomaly.[2)] This result means that we have $\varphi(e_0^2, \ln^{\Lambda^2}/k^2) = 1 + O(e_0^4)$. Using the values $\beta(e^2) = e^4/12\pi^2$ and $\gamma(e^2) = 3e^4/64\pi^4$, we can reproduce Eq. (2).

Without derivation, we offer a generalization of Eq. (2) to the case of a non-Abelian gauge theory with the $SU(N)$ group, with a fermion which belongs to representation R:

$$(\partial_\mu j^5_\mu)_{ren} = F\tilde{F}\frac{g^2(k)}{8\pi^2}\, T(R)\left(1 - \frac{3g^2(k)}{4\pi^2}\,\frac{T(R)C_2(R)}{\frac{11}{3}N - \frac{4}{3}T(R)} + \frac{g^2(k)N}{4\pi^2}\left(1 - \frac{1}{2}k^2 I_{00}\right)\right). \qquad (8)$$

Here I_{00} is given by (4) with $m = 0$. We see that the expression for $\partial_\mu j^5_\mu$ is not a numerical series in $g^2(k)$ in this case. The last term in (8) comes from the gluon self-effect.

We turn now to supersymmetric QED. We have already calculated the radiative corrections to the vortex for the emission of two photons from one point: the quantity $F\tilde{F}$. In a supersymmetric theory, $F\tilde{F}$ is in the imaginary part of the F term of the square of the superfield, $W^\alpha W_\alpha = W^2$. We must accordingly find the radiative corrections to W^2. We consider the generating functional J in the presence of an external vector superfield

$$J = \int DVD\phi_+ D\phi_- \exp iS(V + V_{ext}, \phi_+, \phi_-), \qquad (9)$$

$$S = \frac{1}{4g_0^2}\int d^4x\Big(([W^2]_F + [\bar{W}^2]_F) + [\phi_+^+ e^V\phi_+ + \phi_-^+ e^{-V}\phi_-]_D\Big).$$

The logarithmic derivative of the functional J with respect to $1/g_0^2$ determines the exact value (including all of the radiative corrections) of the quantity W^2, integrated over $d^4xd^2\theta$:

$$\int d^4xd^2\theta\langle W^2\rangle = -4i\partial/\partial(1/g_0^2)\ln J. \qquad (10)$$

The differentiation here should be understood somewhat loosely: The coefficients of W^2 and $\bar{W}^2$ in the expression for the action must be regarded as independent.

On the other hand, we can write J in terms of an effective action which depends on the external field V_{ext} and the characteristic momentum associated with this field:

$$J = e^{iS_{eff}}; \quad S_{eff} = \int d^4x\,\frac{1}{4g^2(k)}\,([W^2_{ext}]_F + \text{H.a.}). \qquad (11)$$

In the expression for S_{eff} we have retained only the terms which are quadratic in W_{ext}. Differentiating (11) with respect to $1/g_0^2$, we find

$$\langle W^2\rangle = \frac{\partial}{\partial(1/g_0^2)}\,\frac{1}{g^2(k)}\,W^2_{ext} = \frac{g_0^4\,\beta(g^2(k))}{g^4\,\beta(g_0^2)}\,W^2_{ext} = \frac{\beta(g^2)\beta_1(g_0^2)}{\beta_1(g^2)\beta(g_0^2)}\,W^2_{ext}, \qquad (12)$$

where β and β_1 are the exact and single-loop β functions. We have tested the validity of (12) in the first approximation in g_0^2 ($\sim g_0^4\ln(\Lambda^2/k^2)$) by a direct calculation in which we made use of the method of supergraphs in a background field.[17] Grisaru *et* *al.*[14] have pointed out that a regularized expression for j_μ^5 can be constructed by various methods. In one of these methods, the quantity $\partial_\mu j_\mu^5$ does not have a two-loop correction, and at this accuracy level the divergence of the current would satisfy the Adler-Bardeen theorem. However, the diagrams corresponding to the scattering of light by light [Eq. 12] make the three-loop correction $\sim g_0^6 \ln(\Lambda^2/k^2)$ nonzero. After a renormalization, i.e., after a multiplication of $\partial_\mu j_\mu^5$ by a suitable $Z(g_0^2)$, we find a correction $\sim g^4(k)$, which agrees with the appearance of a complete β function in the equation for $\partial_\mu j_\mu^5$. In another method for determining the current j_μ^5, it appears in one supermultiplet with an energy-momentum tensor $\theta_{\mu\nu}$. In this case, Grisaru *et al.*[14] found a correction $\sim g_0^4$ even at the two-loop level. That result is not surprising: In this case the current j_μ^5 does not require renormalization (its anomalous dimensionality, like $\theta_{\mu\nu}$, is zero), so there are no functions $Z(g^2(k))$ and $Z(g_0^2)$ in Eq. (6). The function $\varphi(g^2,0)$, however, which depends on the definition of the regularized current, is actually nonzero: $\varphi(g^2(k),0) = \beta(g^2)/\beta_1(g^2)$. The final result for the quantity $\partial_\mu j_\mu^5$ is found by the same method as was used for the first determination of the current.

In a separate paper we will analyze physical applications of the multiloop nature of the anomaly.

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[1)]The quantity j_μ^5 is renormalized multiplicatively. The current K_μ ($\partial_\mu K_\mu = F\tilde{F}$), which has the same dimensionality, cannot be (locally) mixed with j_μ^5, since it is gauge-invariant.

[2)]Novikov *et al.*[13] have linked this fact with the "two-limit technique" which was used in Ref. 1. It can be shown by the background-field method, however, that the two-loop correction actually disappears even before the integration over the coordinates.

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