Formula for a Wilson loop

D. I. D'yakonov and V. Yu. Petrov
B. P. Konstantinov Institute of Nuclear Physics, Academy of Sciences of the USSR

(Submitted 23 January 1989)
Pis'ma Zh. Eksp. Teor. Fiz. **49**, No. 5, 251–254 (10 March 1989)

A new formula is derived for a contour-ordered exponential function from a Yang-Mills field. It can be represented as a functional integral over all gauge transformations of a given field. In the course of the analysis we derive an evolution operator for a Wess-Zumino action.

A contour-ordered exponential function from a Yang-Mills field (we will say simply "*P*-exponential") is frequently used in gauge theory. It is a complicated matter to carry out analytic calculations with a *P*-exponential, however, because it is essentially impossible to calculate in a general form, e.g., in the field of an ensemble of instantons or merons. For this reason alone, we would like to have some alternative representations for it.

Let us consider a *P*-exponential along a given curve $x_\mu(t)$. With the beginning

 0021-3640/89/050284-05$01.00

and end points we associate the parameter values t_1 and t_2. For simplicity we restrict the analysis to the non-Abelian group SU_2. We denote by T^a ($a = 1, 2, 3$) the generators of the SU_2 group in the given representation T, $T^a T^a = T(T+1)$. We denote by $A(t) \equiv A^a_\mu T^a dx_\mu / dt$ the component of the Yang-Mills field which is tangent to the curve. We define the P-exponential

$$W_{\alpha\beta}(t_2, t_1) = [P\exp i \int_{x(t_1)}^{x(t_2)} A^a_\mu T^a dx_\mu]_{\alpha\beta} = [P\exp i \int_{t_1}^{t_2} A(t) dt]_{\alpha\beta}, \qquad (1)$$

as the solution of the equation

$$[i \frac{d}{dt} + A(t)]_{\gamma\alpha} \psi_\alpha(t) = 0:$$

$$\psi_\alpha(t_2) = W_{\alpha\beta}(t_2, t_1) \psi_\beta(t_1). \qquad (2)$$

An expansion of the P-exponential in a power series in the field takes the form

$$W_{\alpha\beta}(t_2, t_1) = \sum_{n=0}^{\infty} \int d\tau_1 \cdots d\tau_n [iA(\tau_1) \cdots iA(\tau_n)]_{\alpha\beta}, \qquad (3)$$

where $t_2 \geqslant \tau_1 \geqslant \cdots \geqslant \tau_n \geqslant t_1$. Along a given curve the potential $A(t)$ can always be represented as a "pure gauge":

$$A_{\alpha\beta}(t) = iD^T_{\alpha\gamma}(U(t)) \frac{d}{dt} D^T_{\gamma\beta}(U^+(t)), \qquad (4)$$

where $D^T(U)$ is the finite rotation matrix (Wigner function) in representation T. In particular, in the spinor representation we have $D^{1/2}_{\alpha\beta}(U) = U_{\alpha\beta}$. It follows from Eq. (2) that P-exponential (1) is

$$W_{\alpha\beta}(t_2, t_1) = D^T_{\alpha\gamma}(U(t_2)) D^T_{\gamma\beta}(U^+(t_1)) = D^T_{\alpha\beta}(U(t_2) U^+(t_1)). \qquad (5)$$

The formula for P-exponential (1) which we are seeking constitutes a functional integral over all gauge transformation $S(t)$ of the field $A(t)$, which is then projected into the matrix representation α, β:

$$W_{\alpha\beta}(t_2, t_1) = \int\int dS_1 dS_2 (2T+1) D^T_{\alpha T}(S_2^+) D^T_{T\beta}(S_1) \int_{S(t_1)=S_1}^{S(t_2)=S_2} \times DS(t) \exp\{iT \int_{t_1}^{t_2} dt \operatorname{Tr} \tau_3 (SAS^+ + i\dot{S}S^+)\}. \qquad (6)$$

Here it is assumed that $A(T)$ is written in the spinor representation: $A = A^a \tau_a / 2$, where τ_a are the Pauli matrices, and $S(t)$ is a 2×2 unitary gauge-transformation matrix. The integrals over $dS_{1,2}$ are normalized integrals over the Haar measure of group SU_2.

Here it is assumed that $A(T)$ is written in the spinor representation: $A = A^a \tau_a/2$, where τ_a are the Pauli matrices, and $S(t)$ is a 2×2 unitary gauge-transformation matrix. The integrals over $dS_{1,2}$ are normalized integrals over the Haar measure of group SU_2.

To prove (6) we assume, without any loss of generality, a "pure gauge" potential [cf. (4)]: $A(t) = iU(t)\dot{U}^+(t)$. The right side of (6) must then have the form of (5). We rewrite the integrand in exponent (6) as $SAS^+ + i\dot{S}S^+ = iR\dot{R}^+$, where $R = \mathrm{SU}$. Making a small shift in the integrals over the Haar measure, $dS = d(\mathrm{SU}) = dR$, we can rewrite (6) as

$$W_{\alpha\beta}(t_2, t_1) = \int\int dR_1 dR_2 (2T+1) D^T_{\alpha T}(U(t_2)R_2^+) D^T_{T\beta}(R_1 U^+(t_1)) Z(R_2, R_1), \quad (7)$$

$$Z(R_2, R_1) = \int_{R_1}^{R_2} DR(t) \exp\{iT\int_{t_1}^{t_2} dt\, \mathrm{Tr}(iR\dot{R}^+\tau_3)\}. \quad (8)$$

Functional integral (8) is the evolution operator of a top with a Wess-Zumino action, but without the ordinary "kinetic term," which is quadratic in time derivatives. The more general problem of the evolution of a symmetric top with a Wess-Zumino action was analyzed in Ref. 1. To make use of the results of that study, we introduce terms in (8) which are quadratic in the time derivatives and which have small moments of inertia I, which we will subsequently let go to zero. We thus consider instead of (8) the evolution operator

$$Z_{Reg}(R_2, R_1) = \int_{R_1}^{R_2} DR(t) \exp\{i\int_{t_1}^{t_2} dt\,[\frac{I_\perp}{2}(\Omega_1^2 + \Omega_2^2) + \frac{I_\parallel}{2}\Omega_3^2 + T\Omega_3]\}, \quad (9)$$

where $\Omega_a = i\mathrm{Tr}(R\dot{R}^+\tau_a)$ are the angular velocities of the top. We write (9) as a sum over intermediate states of the top:

$$Z_{Reg}(R_2, R_1) = \sum_{J,m,k} (2J+1) D^{J*}_{mk}(R_1) D^J_{mk}(R_2) \exp[-i(t_2 - t_1)E_{Jm}], \quad (10)$$

where, according to Eq. (1.34) of Ref. 1, we have

$$E_{Jm} = \frac{J(J+1) - m^2}{2I_\perp} + \frac{(m-T)^2}{2I_\parallel}, \qquad |m| \leqslant J. \quad (11)$$

As the moments of inertia $I_{\perp,\parallel}$ tend toward zero, a single intermediate state, with a lower energy ($m = J = T$), survives in sum (10). The remaining exponential factor from the lower energy can be eliminated through an appropriate normalization of integral (9), since this step corresponds to a general shift of the energy scale. As a results, we find, for the original evolution operator, (8),

$$Z(R_2, R_1) = (2T+1)\sum_k D^{T*}_{Tk}(R_1) D^T_{Tk}(R_2) = (2T+1) D^T_{TT}(R_2 R_1^+). \quad (12)$$

The representation and the projection of the moment are fixed by the coefficient T in front of the Wess-Zumino term in (8), so this evolution operator depends on only the relative orientation of the top at the initial and final times. It does not depend on the time interval $t_2 - t_1$. Result (12) could also be derived directly without difficulty by treating functional integral (8) as the limit of an infinite number of ordinary integrals at discrete points. In this derivation again, however, we would need to carry out a regularization in order to make $R(t)$ smooth at neighboring points, since integral (8) is not of the Feynman type.

Substituting (12) into (7), and integrating over the final matrices $R_{1,2}$, we immediately find an expression for the P-exponential function in the form in (15). Q.E.D.

For a Wilson loop, i.e., for a tracing of the P-exponential over a closed contour, we find a simpler expression, which also follows from (12):

$$\mathrm{Tr}\,P\exp i\oint A_\mu^a T^a dx_\mu = \int DS(t)\exp\{iT\int dt\,\mathrm{Tr}\,\tau_3(SAS^+ + iS\dot{S}^+)\}, \tag{13}$$

where t parametrizes the given closed curve. Note that the matrix τ_3 here could be replaced by any other Pauli matrix. The gauge invariance of (13) is obvious. The fact that the Yang-Mills field appears simply in an exponential function (without an ordering) substantially simplifies the averaging of the Wilson loop in the given ensemble of external fields. We wish to emphasize that the entire dependence on the representation is in the common factor T in the exponential function.

We introduce the unit vector $n_a(t) = (1/2)\mathrm{Tr}S\tau_a S^+\tau_3$. The first term in (13) is rewritten as $A^a n_a$, and the second can be written in the more standard form of a Wess-Zumino term[1]:

$$S_{WZ} = \frac{1}{2}\int d^2t\,\epsilon_{\mu\nu}\,\epsilon_{abc}\,n_a\partial_\mu n_b\partial_\nu n_c, \tag{14}$$

where the integration is carried out over any area inside the contour. The Wilson loop can thus be written in the form

$$\mathrm{Tr}\,P\exp i\oint A_\mu^a T^a dx_\mu = \int D\mathbf{n}(t)\exp iT[\int dt(\mathbf{An}) + S_{WZ}]. \tag{15}$$

In this form, the equation becomes nearly obvious: The P-exponential function, as follows from its definition (2), is the evolution operator of a "spin" $\mathbf{n}$ in a time-varying "magnetic field" $\mathbf{A}(t)$. The Wess-Zumino term specifies the representation to which the given spin belongs.

In the special case $\mathbf{A}(t) = \mathrm{const}$, we find that the integral with the Wess-Zumino action gives us an expression for the character of the representation, T:

$$\int D\mathbf{n}(t)\exp iT[\int dt(\mathbf{An}) + S_{WZ}] = \sum_{m=-T}^{T}\exp im|\mathbf{A}|l, \tag{16}$$

where l is the length of the curve. This expression was recently derived in Ref. 2.

We are indebted to A. Yu. Morozov, P. V. Pobylitsa, and S. L. Shatashvili for useful discussions.

[1]D. I. D'kyakonov and V. Yu. Petrov, Preprint 967, Leningrad Institute of Nuclear Physics, 1984; in: Elementary Particles. (Proceedings of the Twelfth School of the Institute of Theoretical and Experimental Physics, 1984), Vol. 2, Energoatomizdat, Moscow, 1985, p. 50.
[2]A. Alekseev, L. Faddeev, and S. Shatashvilli, to be published in J. Geometry and Physics.

Translated by Dave Parsons