Bosonization and multiloop corrections for the Wess-Zumino-Witten model

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A representation of a Kac-Moody algebra in terms of free fields is discussed.

1. Finding the multiloop correlation functions in a $2D$ conformal theory requires expressing these functions in terms of the correlation functions of some free fields on a Riemann surface. We will, somewhat loosely, call this procedure "bozonization."[1)] So far, bozonization is being studied primarily for minimal models, which reduce to a theory of free scalar fields with values in a circle.[2,3] In order to make the transition to conformal theories of a general type, it is useful to consider the Wess-Zumino-Witten model,[4,5] since all nontrivial conformal theories can apparently be derived from it by means of a Goddard-Kent-Olive projection.[6]

2. For simplicity we begin with the case of the Wess-Zumino-Witten model which is associated with a Kac-Moody algebra $SU(2)_k$. The currents can then be written in terms of the three free fields χ, W, ϕ, of which χ and W are boson 0- and 1-differentials, while ϕ is a scalar field with values in a circle[7]:

$$\begin{aligned} J_+ &= W \\ H &= 2\chi W + \sqrt{2}q\,\partial\phi \\ J_- &= \chi^2 W + \sqrt{2}q\chi\partial\phi + (2-q^2)\partial\chi\ . \end{aligned} \tag{1}$$

The operator expansions are of the form $W(z)\chi(0) = +1/z + ...$, $\phi(z)\phi(0) = +\log z + ...$; $J_\pm(z)H(0) = \pm 2J_\pm/z + ...$, $J_+(z)J_-(0) = -k/z^2 + H/z + ...$, $H(z)H(0) = +2k/z^2 + ...$. The central charge of a Kac-Moody algebra is $k = -2 + q^2$, and the energy-momentum tensor is

$$T = \frac{1}{2(k+2)} : J_+J_- + J_-J_+ - \frac{1}{2}H^2 : = -W\partial\chi - \frac{1}{2}(\partial\phi)^2 - \frac{\sqrt{2}}{2q}\partial^2\phi\ . \tag{2}$$

The central charge is $c = 3k/(k+2) = 3 - 6/q^2$; we have $c_{W\chi} = +2$ and $c_\phi = 1 - 6/q^2$. The radius of the circle in which ϕ takes on values is proportional to q. At integer values of q^2 the model is a rational conformal theory and can be specified by means of the action

$$k\,\mathrm{Tr}\Big[\int_{d^2\xi} |g^{-1}\partial g|^2 + \int_{d^3\xi} (g^{-1}dg)^3\Big]\ . \tag{3}$$

Correlation functions are constructed in this theory from products of the correlation functions of the free fields ϕ and $W\chi$. The correlation functions of the fields ϕ on a

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surface of arbitrary kind are described in Refs. 3 and 8, among other places. They are expressed in terms of Prime bidifferentials and the theta function Θ_{k+2} of level $k+2$ [i.e., with a matrix of periods which is multiplied by $2(k+2)$]. We will not reproduce these familiar equations here.

The correlation functions of the boson fields $\beta^{(j)}$, $\gamma^{(1-j)}$ with spins j and $1-j$ and with an energy-momentum tensor $T_{\beta\gamma} = -j\beta\partial\gamma + (j-1)\gamma\partial\beta$ are less trivial than the correlation functions for Grassmann b,c systems.[8,9] They are constructed in the following way.[10] First, the boson fields β,γ can be represented locally as the products [11] $\beta^{(j)} = \partial\xi^{(0)}e^{-u}$; $\gamma^{(1-j)} = \eta^{(1)}e^{+u}$, where u is a boson field with values in a circle, and ξ,η are *anticommuting* fields with spins 0 and 1. Here $T_{\xi\eta} = \eta\partial\xi$; $T_u = -\frac{1}{2}(\partial u)^2 - (2j-1)\partial^2 u$. We then have[10]

$$\langle \xi(x_0)\dots\xi(x_n)\eta(y_1)\dots\eta(y_n)e^{a_1u(z_1)}\dots e^{a_mu(z_m)}\rangle(\det\bar{\partial}_0)^{-1/2}$$

$$\sim \frac{\prod_{i<j} E(x_i,x_j)\prod_{i<j} E(y_i,y_j)\delta(\Sigma a-(2j-1)(p-1))}{\prod E(x_i,y_j)\prod_{k<l} E(z_k,z_l)^{a_ka_l}\prod\sigma(z_k)^{(2j-1)a_k}}$$

$$\times\frac{\prod_{i=1}^{n}\theta(-y_i+\Sigma x-\Sigma y+\Sigma az-(2j-1)\Delta_*)}{\prod_{i=0}^{n}\theta(-x_i+\Sigma x-\Sigma y+\Sigma az-(2j-1)\Delta_*)} \tag{4}$$

[$E(x,y)$ is a Prime bidifferential, $\sigma(z)$ is the cross section for the trivial stratification, and θ is the ordinary theta function on a Riemann surface[10]]. In our case we have $j=1$, $\beta^{(1)} = W$, and $\gamma^{(0)} = \chi$.

These expressions are our starting point for constructing conformal blocks in the Wess-Zumino-Witten model $SU(2)_k$. The conformal blocks themselves are certain linear combinations of the correlation functions written above (included in the linear combinations are integrals over noncontractible contours of operators of unit dimensionality; for the case of kind 0 these integrals reduce to generalized hypergeometric functions[2,3]).

3. It is a simple matter to generalize this construction to the case of a Wess-Zumino-Witten model with an arbitrary algebra G or its Goddard-Kent-Olive projection. The starting point for constructing a bosonization of the type in (1) is a representation of group G in the algebra of vector fields on the space G/H. In this case χ_α ($\alpha = 1,\dots,\dim_C G/H$) are coordinates on G/H, and we have $W_\alpha \sim \partial/\partial\chi_\alpha$. To derive (1), we must allow W_α and χ_α to depend on z, $\bar{z}$, and we must construct a central expansion.

The Wess-Zumino-Witten model itself with a central charge $c = kD/$

$(k+C_v)=D(1-C_v/(k+C_v)$ corresponds to the choice of a homogeneous space G/H of dimensionality $D-r$ (D and r are the dimensionality and rank of algebra G). The independent free fields χ_α and W_α (which are, as before, boson 0- and 1-differentials) are then associated with positive roots of the Lie algebra G, α, $\in$, Σ_+, while the scalar fields ϕ_i $(i=1,...,r)$ are associated with values in circles: Cartan generators. The current corresponding to the Cartan vector $\vec{\mu}$ has the simple form

$$H_\mu = \sum_{\alpha\in\Sigma_+} \langle \mu,\alpha\rangle \chi_\alpha W_\alpha + q\vec{\mu}\partial\vec{\phi}. \tag{5}$$

The central charge of the algebra is thus

$$k = -C_v + q^2, \tag{6}$$

($\langle\mu,\nu\rangle C_v \equiv \Sigma_{\alpha\in\Sigma_+}\langle\mu,\alpha\rangle\langle\alpha,\nu\rangle$), and the energy-momentum tensor of the fields χ_α, W_α, and ϕ_i is

$$T = \frac{1}{2(k+C_v)}:\sum_{\alpha\in\Sigma_+}\left(J_\alpha J_{-\alpha} + J_{-\alpha}J_\alpha - \frac{1}{C_v}H_\alpha H_\alpha\right): = -\sum_{\alpha\in\Sigma_+} W_\alpha\partial\chi_\alpha - \frac{1}{2}(\partial\vec{\phi})^2$$
$$-\frac{1}{2q}\sum_{\alpha\in\Sigma_+}\vec{\alpha}\,\partial^2\vec{\phi}\,. \tag{7}$$

The central charge $c=D-DC_v/(k+C_v)$ is the sum of $c_{W_\alpha,\chi_\alpha}=+2$, of $c_{\vec{\phi}}=+1$ for all $\vec{\phi}$ which are orthogonal with respect to the vector $\vec{\rho}\equiv\frac{1}{2}\Sigma_{\alpha\in\Sigma_+}\vec{\alpha}$, and of the central charge of the field $\vec{\phi}_A$, which is collinear with this vector, which is equal to $1-12\vec{\rho}^2/q^2=1-DC_v/(k+C_v)$. The 1-loop characters are $\chi\sim\eta(q)^D(\theta_{k+C_v})^r\theta^{D-r}$.

The other currents, which are Kac-Moody generators, $j_{\pm\alpha}$ for $\alpha\in\Sigma_+$ are given by rather lengthy expressions. For example, in the $SU(3)_k$ case (Fig. 1) we have

$$J_1 = W_1 - a\chi_3 W_2 \tag{8}$$

$$J_2 = W_2$$

$$J_3 = W_3 + b\chi_1 W_3$$

$$J_{-1} = \chi_1^2 W_1 + b\chi_1\chi_2 W_2 - b\chi_1\chi_3 W_3 + \chi_2 W_3 + ab\chi_1^2\chi_3 W_2 + (2+b-q^2)\partial\chi_1 + q\chi_1\vec{\alpha}_1\partial\vec{\phi}$$

$$J_{-2} = \chi_1\chi_2 W_1 + \chi_2^2 W_2 + \chi_2\chi_3 W_3 + a\chi_1^2\chi_3 W_1 - b\chi_1\chi_3^2 W_3 + ab\chi_1^2\chi_3^2 W_2 + (3-q^2)\partial\chi_2 + q\chi_2\vec{\alpha}_2\partial\vec{\phi} + a(2-q^2)\chi_3\partial\chi_1 - b(2-q^2)\chi_1\partial\chi_3 + q\chi_1\chi_3(a\vec{\alpha}_1 - b\vec{\alpha}_3)\partial\vec{\phi}$$

$$J_{-3} = -a\chi_1\chi_3 W_1 + a\chi_2\chi_3 W_2 + \chi_3^2 W_3 - \chi_2 W_1 - ab\chi_1\chi_3^2 W_2 + (2+a-q^2)\partial\chi_3 + q\chi_3\vec{\alpha}_3\partial\vec{\phi}$$

$$H_1 = 2\chi_1 W_1 + \chi_2 W_2 - \chi_3 W_3 + q\vec{\alpha}_1\partial\vec{\phi}$$

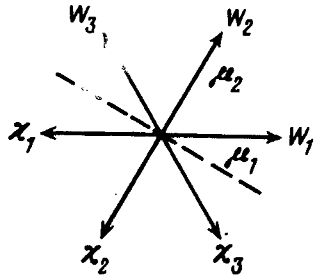

FIG. 1. Diagram of the roots of the $SU(3)$ algebra.

$$H_2 = \chi_1 W_1 + 2\chi_2 W_2 + \chi_3 W_3 + q\vec{\alpha}_2 \partial \vec{\phi}$$

$$H_3 = -\chi_1 W_1 + \chi_2 W_2 + 2\chi_3 W_3 + q\vec{\alpha}_3 \partial \vec{\phi};$$

$$J_{\vec{\alpha}}(z) J_{\vec{\beta}}(0) = N_{\vec{\alpha},\vec{\beta}} J_{\vec{\alpha}+\vec{\beta}}/z + \dots \quad \text{for} \quad \vec{\alpha}+\vec{\beta} \in \Sigma \ (N_{\vec{\alpha},\vec{\beta}} = \pm 1),$$

$$J_\alpha(z) H_\beta(0) = \langle \alpha, \beta \rangle J_\alpha / z + \dots ,$$

$$J_\alpha(z) J_{-\alpha}(0) = -k/z^2 + H_\alpha/z + \dots, \quad H_\alpha(z) H_\beta(0) = +\langle \alpha, \beta \rangle k/z^2 + 0(1);$$

$$T = -W_1 \partial \chi_1 - W_2 \partial \chi_2 - W_3 \partial \chi_3 - \frac{1}{2}(\partial \phi_1)^2 - \frac{1}{2}(\partial \phi_2)^2 - \frac{\sqrt{2}}{q} \partial^2 \phi_2 . \tag{9}$$

The central charges are $k = -3 + q^2$ and $c = 8k/(k+3) = 8 - 24/q^2$. The vectors $\vec{\alpha}_1$, $\vec{\alpha}_2$, $\vec{\alpha}_3$ are directed along the three positive roots and are of identical length $\sqrt{\langle \alpha,\alpha \rangle} = \sqrt{2}$. The parameters a and b are related by

$$a + b = 1. \tag{10}$$

The specific choice of a is not important [representation (8) simplifies slightly if we choose $a = 0$ and $b = 1$]. Equations (8) correspond to the space $G/H = SU(3)/U(1) \times U(1)$. Other choices of G/H are associated with Goddard-Kent-Olive projections of the $SU(3)$ model. The number of fields χ_α, W_α decreases in the obvious way.

All of the fields χ_α, W_α, and ϕ_i are free and independent. The correlation functions of these fields which we need for constructing the conformal blocks are described in §2. Note, however, that—as in the case of minimal models—choosing correct linear combinations is not a totally trivial matter [and this is true even in the case of the free $SU(2)_{k=1}$ theory].

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[1)]In the literature this term is sometimes used in a more general sense: in a reformulation of the theory in terms of boson fields which are not free (the energy-momentum tensor is not quadratic) (Ref. 1, for example). Such a formulation, however, does not make it possible to calculate correlation functions directly.

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