Gapless fermion excitations at vortices in superfluids and superconductors

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A quantized vortex in a superfluid or superconducting Fermi system has gapless fermion excitations which are localized at the core of the vortex. The number of branches of gapless excitations—the number of fermion zero modes—varies from 0 to $\epsilon_F/\Delta \gg 1$, depending on the structure of the vortex core.

A superfluid Fermi liquid and a superconductor have low-lying fermion excitations which are localized at the cores of quantized vortices.[1] In conventional *s*-pairing superconductors, excitations which are localized near the axis of a vortex, where the phase of the order parameter has a singularity, and where the modulus of the order parameter vanishes, have an energy gap $\sim \Delta^2/\epsilon_F$ which is small in comparison with the gap Δ of unlocalized fermions. These excitations accordingly play a governing role

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in the thermodynamics and kinetics of the system at low temperatures (see the review by Gor'kov and Kopnin[2]).

Modern field theories also discuss vortices (strings) with fermions localized in their cores (e.g., Ref. 3). An important point is that in most models one or several excitations are so-called zero modes; they do not have a gap. The number of fermion zero modes in these models is related by the index theorem to the topological charge of the vortex, i.e., to the phase shift when the axis of the vortex is circumvented.

Since the presence or absence of a gap for localized fermions leads to different types of behavior of a system at sufficiently low temperatures, $T \ll \Delta^2/\epsilon_F$, it is necessary to determine the conditions under which gapless fermions appear. In the present letter we show that a necessary condition for the appearance of zero modes in a system with Cooper pairing is a spreading out of the singularity in the phase at the axis of the vortex. A spreading out of this sort occurs in vortices in superfluid ^{3}He-B, where a vortex with a singularity at the axis (an o vortex) is unstable with respect to the formation of a v vortex with broken parity, in which there is no singularity at the axis, and the order parameter vanishes nowhere (see the review in Ref. 4). It also occurs in two-quantum Abrikosov vortices in ordinary superconductors.[5] Nonsingular vortices are distinguished from singular vortices by the behavior of semiclassical fermion spectrum, $E(\mathbf{k},\mathbf{r}) = \sqrt{\epsilon^2(\mathbf{k}) + |\Delta(\mathbf{k},\mathbf{r})|^2}$. In a singular vortex, the energy $E(\mathbf{k},\mathbf{r})$ vanishes at the axis of the vortex ($r=0$) over the entire Fermi surface ($|\mathbf{k}| k_F$), while in a nonsingular vortex for each point $\mathbf{r}$ within a certain radius r_0 the energy vanishes at four points on the Fermi surface: $\mathbf{k} = \mathbf{k}^a(\mathbf{r})$, where $a = 1$–4. These are exceptional "diabolical points" of the spectrum,[6] which are not removable, because of the conservation of the topological invariant. They are also known as "boojums on the Fermi surface."[7]

It turns out that the number of zero modes at vortices in systems with Cooper pairing depends not on the topological charge of the vortex but on the spatial distribution $\mathbf{k}^a(\mathbf{r})$ of the zeros in the semiclassical spectrum of fermions. This number can, for a given phase shift, vary from 0 to ϵ_F/Δ.

The number of branches of gapless Bogolyubov excitations is determined by the index $N(k_z)$ of the Bogolyubov operator, as a function of the conserved quantum number: the momentum along the vortex axis, k_z. The integer index $N(k_z)$ determines half the difference between the numbers of negative and positive fermion energy levels at the given k_z, so a unit change in N at some k_z means that at this k_z the energy level crosses zero. The index N can be expressed in terms of the Green's function $\hat{G} = (i\omega - \hat{H})^{-1}$:

$$N(k_z) = \mathrm{Tr} \int \frac{d\omega}{2\pi} \hat{G} = - \mathrm{Tr} \int \frac{d\omega}{2\pi} \frac{\hat{H}}{\omega^2 + \hat{H}^2} = -\frac{1}{2} \sum_n \operatorname{sign} E_n(k_z), \tag{1}$$

where Tr is the sum over all states of the Bogolyubov Hamiltonian $\hat{H}$ which have the given k_z. Since the size of the vortex core is on the order of the coherence length $\xi \sim (v_F/\Delta) \gg k_F^{-1}$, we can use a gradient expansion for the Green's function[7]:

$$N(k_z) = \frac{1}{2}\int \frac{d^2k_\perp d^2r}{(2\pi)^2}\left(\frac{\partial}{\partial \mathbf{r}}\Phi\frac{\partial}{\partial \mathbf{k}}n - \frac{\partial}{\partial \mathbf{r}}n\frac{\partial}{\partial \mathbf{k}}\Phi\right) = \frac{1}{2}\int \frac{d^2k_\perp d^2r}{(2\pi)^2} n\left(\frac{\partial}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{k}} - \frac{\partial}{\partial \mathbf{k}}\frac{\partial}{\partial \mathbf{r}}\right)\Phi, \tag{2}$$

where $n(\mathbf{k},\mathbf{r}) = \frac{1}{2} - \frac{1}{2}[\epsilon(\mathbf{k})/E(\mathbf{k},\mathbf{r})]$ and $\Phi(\mathbf{k},\mathbf{r})$ are respectively the number of particles and the phase of the determinant of the spin matrix of the order parameter in the semiclassical approximation. The second equation in (2) was found through an integration by parts. We thus see that $N(k_z)$ may be nonzero if there are nonremovable singularities in $\Phi(\mathbf{k},\mathbf{r})$, i.e., diabolical points $\mathbf{k}^a(\mathbf{r})$ in the semiclassical spectrum. According to Ref. 7, the difference between the mixed derivatives is expressed in terms of the distribution of boojums in the following way:

$$n\left(\frac{\partial}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{k}} - \frac{\partial}{\partial \mathbf{k}}\frac{\partial}{\partial \mathbf{r}}\right)\Phi = 2\pi\sum_a N^a\mathbf{k}^a \,\mathrm{curl}\; \mathbf{k}^a \int_0^1 du u\delta(\mathbf{k} - u\mathbf{k}^a(\mathbf{r})). \tag{3}$$

As a result, we find the following expression for the spectral asymmetry:

$$N(k_z) = -\frac{1}{4\pi}k_z \sum_a N^a \int d^2r k_z^a |k_z^a|^{-3}(\mathbf{k}^a \,\mathrm{curl}\; \mathbf{k}^a)\theta\left(\frac{k_z^a(r)}{k_z} - 1\right), \tag{4}$$

where N^a is an integer topological charge of the a-th diabolical point.

In the case of ^{3}He-A, in which two paired diabolical points are present even in a homogeneous state and are expressed in terms of the single orbital vector $\mathbf{1}$ ($\mathbf{k}^1 = k_F\mathbf{1}$, $\mathbf{k}^2 = -k_F\mathbf{1}$, $N^1 = -N^2 = 2$), the spectral asymmetry is

$$N(k_z) = -\frac{k_z}{\pi}\int d^2r l_z |l_z|^{-3}\mathbf{1}\,\mathrm{curl}\;\mathbf{1}\theta(k_F^2 l_z^2 - k_z^2). \tag{5}$$

As a result, the index $N(k_z)$ is nonzero for only those nonsingular vortices in ^{3}He-A which have a twisting texture, i.e., for which $\mathbf{1}\mathrm{curl}\mathbf{1} \neq 0$. These are so-called ω-vortices with broken parity; the $\mathbf{1}$ texture of these vortices is of the form $\mathbf{1}(r) = \hat{\mathbf{z}}\cos\eta(r) + \overline{\varphi}\sin\eta(r)$, where z,r,φ are the cylindrical coordinates, and $\eta(0) = 0$ and $\eta(\infty) = \pi$. These vortices are the ones with the lowest energy.[4] Figure 1 shows the qualitative behavior $N(k_z)$ for a ω-vortex. At $|k_z| > k_F$ we have an index $N(k_z) = 0$; i.e., the numbers of positive and negative levels are the same. In the case $|k_z| < k_F$ there is an asymmetry of levels; i.e., some of the $E_n(k_z)$ branches cross zero. The number of zero modes in a vortex corresponds to the maximum value $N_{\max} \sim k_F R$, where R—the radius of the vortex core—is a length scale of the variation in the $\mathbf{1}$ texture. There are thus $\sim k_F R$ one-dimensional fermions which are moving along the axis of the vortex and which have zero-dimensional Fermi surfaces: points in which the $E_n(k_z)$ spectrum crosses zero. The net momentum k_z of these fermions below the zero-dimensional Fermi surfaces [the dashed part of the $E_n(k_z)$ spectrum in Fig. 1b] leads to a spontaneous mass current along the axis of a ω-vortex:

$$j_z = \frac{1}{2}\int \frac{dk_z}{2\pi}k_z N(k_z) = -\frac{k_F^3}{6\pi^2}\int d^2r(\mathbf{1}\hat{\mathbf{z}})(\mathbf{1}\,\mathrm{curl}\;\mathbf{1}), \tag{6}$$

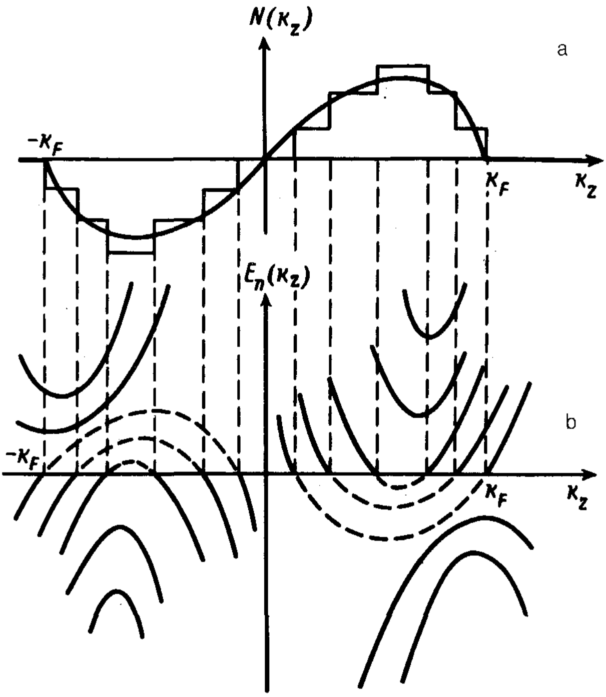

FIG. 1. Qualitative behavior of the spectrum asymmetry function $N(k_z) = -\frac{1}{2}\Sigma_N \operatorname{sign} E_n(k_z)$ and of the branches $E_n(k_z)$ of the one-dimensional spectrum of fermions which are localized in the core of a quantized vortex with a diffuse singularity. The thin stepped line (in part a) reflects the fact that $N(k_z)$ takes on integer values; the heavy curve corresponds to the envelope found by the approximate method of a gradient expansion. At those momenta k_z where $N(k_z)$ is discontinuous, one of the branches $E_n(k_z)$ (in part b) crosses the energy zero. The dashed lines (in part b) show those parts of the spectrum which have a net momentum k_z and which lead to a spontaneous flux of mass (or spin) along the axis of the vortex.

This current corresponds to an anomalous texture current.[8]

Diabolical points also arise in a vortex core in ^{3}He-B. The vortices in ^{3}He-B are of the v type,[4] for which there is typically no mass current along the axis, but there is a spontaneous spin flux. The index $N(k_z)$, given by (1), vanishes by virtue of the same symmetry as is responsible for the vanishing of the mass current. In this case, however, the result does not mean that there are no zero modes. The explanation is that the index $N(k_z)$ is insensitive to the spin of the particles and does not change if the branches $E_{n\uparrow}$ and $E_{n\downarrow}$ of particles with opposite spins cross zero simultaneously, but in different directions. In order to "catch" zero modes even in this case, we have to introduce indices $N_\uparrow(k_z)$ and $N_\downarrow(k_z)$, as was done in Ref. 9. Although the spin projection of the excitations is not conserved in ^{3}He-B, a calculation of the quantity $N_\uparrow(k_z) = \operatorname{Tr} \int (d\omega/2\pi)(1+\sigma_z/2)\widehat{G}$ leads to a correct estimate of the number of zero modes, $k_F R \sim \epsilon_F/\Delta$. Here the circumstance that the radius of the vortex core in ^{3}He-B is on the order of ξ has been taken into account. The net momenta and spins of the zero modes lead to a spontaneous spin current along the axis of a vortex of the v type:

$$j_z^z = \frac{\hbar}{2} \frac{1}{2} \int \frac{dk_z}{2\pi} k_z (N_\uparrow(k_z) - N_\downarrow(k_z)) , \qquad (7)$$

in the absence of a mass current.

We thus see that, in contrast with ordinary singular vortices, vortices with a diffuse singularity may have gapless fermion excitations. In contrast with the field-theory models of fermions on strings, the number of fermion zero modes (the number of "generations") on vortices in condensed media is not determined by the topological charge of the vortex. It is on the order of $k_F R$, where R is the size of the region inside the core in which the diabolical points arise in the semiclassical spectrum $E(\mathbf{k},\mathbf{r})$ as a result of the spreading of the singularity at the vortex axis. The number of generations of fermions is an additional characteristic of vortices which have an identical phase shift and an identical symmetry. Transitions between vortices with different numbers of zero modes are analogous to Lifshitz transitions at $T=0$, for which a zero-dimensional Fermi surface forms or disappears for one-dimensional fermions which are localized at vortices.

A spreading of a singularity at a vortex occurs not only in systems with a multicomponent order parameter such as ^{3}He and, possibly, heavy-fermion superconductors. Singularities also arise in a two-quantum Abrikovsov vortex in an ordinary superconductor[5] and, to a lesser extent, in a single-quantum vortex.[10] Accordingly, we do not rule out the possibility that zero modes will also appear in an ordinary superconductor. The number of these modes may vary from 0 to ϵ_F/Δ.

[1]C. Caroli, P. G. de Gennes, and J. Matricon, Phys. Lett. **9**, 307 (1964).
[2]L. P. Gor'kov and N. B. Kopnin, Usp. Fiz. Nauk **116**, 413 (1975) [Sov. Phys. Usp. **18** 496 (1975)].
[3]E. Witten, Nucl. Phys. **B249**, 557 (1985).
[4]M. M. Salomaa and G. E. Volovik, Rev. Mod. Phys. **59**, 533 (1987).
[5]G. E. Volovik, J. Phys. **C21**, L221 (1988).
[6]P. G. Grinevich and G. E. Volovik, J. Low Temp. Phys **72**, 371 (1988).
[7]G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor. Fiz. **83**, 1025 (1982) [Sov. Phys. JETP **56**, 579 (1982)].
[8]A. V. Balatskiĭ, G. E. Volovik, and V. A. Konyshev, Zh. Eksp. Teor. Fiz. **90**, 2038 (1986) [Sov. Phys. JETP **63**, 1194 (1986)].
[9]A. V. Balatskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **47**, 647 (1988) [JETP Lett. **47**, 746 (1988)].
[10]M. M. Salomaa and G. E. Volovik, J. Phys. **1**, 277 (1989).

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