Torons, projections, Riemann surfaces, and fermion condensates in supersymmetric sigma models

A. Morozov and A. Roslyĭ
Institute of Theoretical and Experimental Physics

(Submitted 20 February 1989)
Pis'ma Zh. Eksp. Teor. Fiz. **49**, No. 7, 365–368 (10 April 1989)

Analogies between instanton solutions and fractional topological charges are discussed.

Because of the summation over all types of boundary conditions, all possible mappings of Riemann surfaces into the factors of the sigma-model manifold **M** under group Γ contribute to the generating functionals of two-dimensional $N=2$ supersymmetric sigma models. If the fields satisfy the classical equations of motion, these manifolds describe nonperturbative fluctuations: instantons, torons, etc. For some of these classical solutions there are precisely two fermion zero modes, so the condensates $\langle\bar{\psi}\psi\rangle$ which arise in $N=2$ supersymmetric sigma models are saturated. A very simple example of this phenomenon, for which M/Γ are nonsingular manifolds without a boundary—flag spaces $M=F_n=SU(n+1)/U(1)^n$—has been studied.

1. Two-dimensional $N=2$ supersymmetric sigma models $L=\int K(\Phi)d^2zd^2\theta d^2\bar{\theta}$ are constructed from Kähler manifolds M with a Kähler potential K. In all sigma models on compact Kähler manifolds there are instantons which are solutions of the duality equations $\bar{\partial}\phi=0$. The number (n_F) of real fermion zero modes in the field of an instanton is determined by the index theorem:

$$n_F=\frac{2}{\pi}\int R_{m\bar{n}}\partial\phi^n_{inst}\overline{\partial\phi^n_{inst}}d^2z.$$

An instanton calculus in supersymmetric theories[1] has made it possible, in particular, to show that the existence of fermion condensates $\langle(\bar{\psi}\psi)^{n_F/2}\rangle$ follows from the existence of solutions of the duality equations with n_F fermion zero modes. The correlation decay principle, if it holds for such supersymmetric theories, would then mean that there must also exist binary condensates $\langle\bar{\psi}\psi\rangle$. A classical solution with $n_F=2$ would have led to the formation of just this type of condensate. So far, a solution of this sort is known only for the $SU(N)$ supersymmetric Yang-Mills model with $d=4$ (Ref. 2). These are the well-known 't Hooft torons.[3] In 1984, A. S. Shvarts and the present authors (unpublished) found a solution with $n_F=2$ for a two-dimensional O_3-sigma model [$M=SU(2)/U(1)$ two-dimensional sphere]—a so-called projection (more on this below).

2. The general formulation of the problem of a nonperturbative quantum-mechanical description of a theory with a given classical Lagrangian presupposes a summation over all the boundary conditions at the space-time infinity. Here the physical fields can take on values in the minimal of the manifolds of fields, which is compatible with the symmetries of the original action. One should thus consider all possible

mappings $S^{(d)} \to M/\Gamma$. In conformally invariant theories, the $S^{(d)}$ are any conformally planar d-dimensional manifolds; for two-dimensional sigma models, the $S^{(2)}$ are any Riemann surfaces, possibly with an edge or with singularities. Here M is the manifold in which the fields take on values, in this case a sigma-model manifold; Γ is some invariance subgroup of the classical action. Various topological mapping classes, which differ in the particular choice of $S^{(d)}$ and Γ, contribute to various correlation functions. Binary fermion correlation functions are determined by mappings with $n_F = 2$.

In some cases it is apparently reasonable to understand M as representing an entire equivalence class of sigma-model manifolds. In cases with discrete groups Γ the factor M/Γ may turn out to be a singular manifold: an orbifold. This happens even in the case[1] $M = CP^n$. In the simplest cases, Γ acts freely—without fixed points—on M, and M/Γ is an ordinary nonsingular manifold without an edge (but which is possibly nonorientable). We will discuss one such example below: that with $M = F_n = SU(n+1)/U(1)^n$.

3. We begin with the simplest version of an O_3 sigma model: $M = F_1 = SU(2)/U(1) = CP^1 =$ two-dimensional sphere. This example is particularly simple because the dimensionalities of manifold M and the space-time are the same. If we impose the boundary conditions $\phi(z) \to \text{const}$ as $|z| \to \infty$, which correspond to a sphere, at the space-time infinity, the mapping $CP^1 \to F_1$ with the minimal topological charge—an instanton—describes a field configuration with four real fermion zero modes: $n_F = 4$. A mapping with $n_F = 2$ must be "half as large": $CP^1/Z_2 \to F_1/Z_2$. The space-time CP^1/Z_2 cannot have singularities, at least not at any finite points, so in this case we would like to have Z_2 act on CP^1 without fixed points. There is such a Z_2 isometry for CP^1: In complex coordinates Z on CP^1, it acts in accordance with the rule $z \to -1/\bar{z}$. Fixed points would have to satisfy the quality $|z|^2 = -1$, so they do not exist. The factor space CP^1/Z_2 is a nonorientable Riemann surface without an edge: a real projective space RP^2. The mapping of the space-time CP^1/Z_2 into a sphere F_1, which has a unit topological charge in the ordinary sense, is not a minimal mapping if we regard it as a mapping into the manifold $RP^2 = F_1/Z_2$. The minimal mapping $CP^1/Z_2 \to F_1/Z_2$ has a topological charge of 1/2 and leads to half as many zero modes: $n_F = 2$.

Note also that a factorization in accordance with an antiholomorphic Z_2 isometry does not disrupt the duality equations. A splicing of different maps (nonholomorphic) is of course matched for the identity mapping $CP^1/Z_2 \to F_1/Z_2$. The "projection" constructed in this fashion solves the problem of the classical configuration with $n_F = 2$ in the case of a sphere. To show that a toron would be unsuitable for this purpose, it is sufficient to verify that any mapping of a torus $R^2/A_1 \times Z_1 \to F_1/Z_2$ (for any imaginable Z_2 isometry) is continued to the mapping $R^2/Z_1 \times Z_1 \to F_1$ and thus has an integer topological charge.

4. In the general case of a sigma model on a manifold F_n we have $n_F = 4$ for an instanton solution,[7] so for our purposes we need a configuration with half the topological charge. In order to construct a corresponding solution we must find (a) a Z_2 symmetry of F_n and (b) a mapping of a Riemann surface into F_n/Z_2 which is not continued to a mapping into all F_n. We will show that a projection solves the problem for all n.

For example, the Kähler manifold $F_2 = SU(3)/U(1)\times U(1)$ is embedded as a complex submanifold in the product $CP^2\times CP^2$. The space $CP^2 = SU(3)/SU(2)\times U(1)$ can be written in terms of the uniform coordinates p_1,p_2,p_3. Let us assume that q_1,q_2,q_3 are uniform coordinates on the second CP^2. We can then specify F_2 in $CP^2\times CP^2$ by the analytic equation $p_i q_i = 0$. On the product $CP^2\times CP^2$, the group $SU(3)\times SU(3)$ acts transitively; the equations $p_i q_i = 0$ and thus F_2 are invariant under the diagonal subgroup $SU(3)$. The invariance group of an individual pont F_2 is $U(1)\times U(1)$; for $p_2 = p_3 = q_1 = q_3 = 0$, say, the meaning is rotations of the phases of the numbers $p_1 \neq 0$ and $q_2 \neq 0$. The free action of the discrete antiholomorphic Z_2 isometry on F_2 can be defined by

$$\begin{pmatrix} p_1 \to +\bar{p}_2 \\ p_2 \to -\bar{p}_1 \\ p_3 \to +\bar{p}_3 \end{pmatrix} \qquad \begin{pmatrix} q_1 \to +\bar{q}_2 \\ q_2 \to -\bar{q}_1 \\ q_3 \to +\bar{q}_3 \end{pmatrix}.$$

This transformation does not alter the equation $p_i q_i = 0$. In $CP^2\times CP^2$ it has fixed points: It follows from

$$\begin{aligned} p_1 &= \lambda\bar{p}_2 & q_1 &= \mu\bar{q}_2 \\ p_2 &= -\lambda\bar{p}_1 & q_2 &= -\mu\bar{q}_1 \\ p_3 &= \lambda\bar{p}_3 & q_3 &= \mu\bar{q}_3 \end{aligned}$$

with arbitrary complex λ and μ only that $p_1 = p_2 = q_1 = q_2 = 0$. These points, however, are not part of submanifold F_2: For them the equation $p_i q_i = 0$ would have taken the form $p_3 q_3 = 0$, and p_3 or q_3 would have had to vanish also, which would be impossible (all of the uniform coordinates cannot be zero simultaneously). The solution of the problem of a mapping with $n_F = 2$ is again given by a projection: $CP^1/Z_2 \to F_2/Z_2$. If $u_1,u_2 (z = u_1/u_2)$ are uniform complex coordinates in the space-time CP^1, this mapping can be written

$$\begin{aligned} p_1 &= u_1 & q_1 &= -u_2 \\ p_2 &= u_2 & q_2 &= u_1 \\ p_3 &= 0 & q_3 &= 0\,. \end{aligned}$$

It is not difficult to see that this mapping is not continued to the mapping $CP^1/Z_2 \to F_2$, so it has half the topological charge, and the corresponding value is $n_F = 2$.

To analyze an arbitrary manifold F_n, it is convenient to use a different representation: in terms of the upper triangular complex $(n+1)\times(n+1)$ matrices V: $V_{ii} = 1, V_{ij} = 0$ for $i > j$ (Ref. 7, for example). A Kähler potential is constructed from the principal minors $\Delta_1,\ldots,\Delta_n$ of the matrix VV^+: $K = \Sigma_{i=1}^n \ln \Delta_i$. The symmetry of the metric and the manifold F_n and the actions of the sigma model constitute a transformation under which K changes into a holomorphic or antiholomorphic function when, for example, the Δ_i are multiplied by the square moduli of analytic functions of elements of V. An antiholomorphic Z_2 symmetry on F_n can be specified in the following way, for example: $V \to (A\bar{V}B)'$. Here the matrix A differs from the unit matrix by virtue of the element $A_{n,n} = +1/\overline{V_{n,n+1}}$, and B by virtue of the replacement of the 2×2 block in the lower right corner by $i\sigma_2$. Multiplication by the matrix B interchanges the last two columns and multiplies one of them by -1. The prime means an interchange of the last two elements of the lower row. For $n=1$, for example, we have

$$V=\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix}\to(A\bar{V}B)'=\left[\begin{pmatrix}1/\bar{\zeta} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \bar{\zeta}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right]'=\begin{pmatrix}1 & -1/\bar{\zeta}\\ 1 & 0\end{pmatrix}'=\begin{pmatrix}1 & -1/\bar{\zeta}\\ 0 & 1\end{pmatrix}.$$

An operation marked with a prime understandably does not affect the first n principal minors $(n+1)\times(n+1)$ of the matrix VV^+, so we can ignore it. Under this transformation we have $VV^+\to A\bar{V}BB^+\bar{V}^+A^+=A\,\overline{VV^+}A$ and $\Delta_i\to\bar{\Delta}_i=\Delta_i$ for $i=1,...,n-1$, and we have $\Delta_n\to(1/|V_{n,n+1}|^2)\Delta_n$, so that this is indeed an isometry. This transformation sends the element $V_{n,n+1}$ into $-1/\overline{V_{n,n+1}}$, so there are no fixed points. Understandably, a projection specified by the formula $V_{n,n+1}=z$ (and for which we otherwise have $V_{ij}=0$, $i\neq j$) is the configuration with $n_F=2$ which we are seeking. For clarity we will write out the most explicit expressions for an antiholomorphic transformation in the cases $n=2,3$;

$$F_2:\begin{pmatrix}1 & \xi & \eta\\ 0 & 1 & \zeta\\ 0 & 0 & 1\end{pmatrix}\to\begin{pmatrix}1 & \bar{\eta} & -\bar{\xi}\\ 0 & 1 & -1/\bar{\zeta}\\ 0 & 0 & 1\end{pmatrix};\quad F_3:\begin{pmatrix}1 & \alpha & \beta & \gamma\\ 0 & 1 & \xi & \eta\\ 0 & 0 & 1 & \zeta\\ 0 & 0 & 0 & 1\end{pmatrix}\to\begin{pmatrix}1 & \bar{\alpha} & \bar{\gamma} & -\bar{\beta}\\ 0 & 1 & \bar{\eta} & -\bar{\xi}\\ 0 & 0 & 1 & -1/\bar{\zeta}\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

5. We have thus shown that the problem of binary fermion condensates in $N=2$ supersymmetric sigma models on flag manifolds can be solved within the framework of a very simple generalization of the "toron idealogy" by an analysis of projections. It has not been necessary to introduce singular factor manifolds. We have not carried out a detailed analysis of other sigma models. In the general case, it would clearly not be sufficient to restrict an analysis to simply projections and nonsingular manifolds. However, there can hardly be any doubt that a solution can be found by this approach for each specific case.

We are indebted to M. A. Shifman, who continuously stimulated our interest in the problem of fermion condensates, and to A. R. Zhitnitskiĭ and A. S. Shvarts for discussions.

[1)]This example was recently analyzed by Zhitnitskiĭ.[4] From the technical standpoint, the analysis of this example is a particular case of string theory on orbifolds (Ref. 5, for example). Another particular case of this theory is an analysis of vortex mappings.[6]

[1]A. I. E. Vaĭnshteĭn, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Fiz. Elem. Chastits At. Yadra **17**, 472 (1986) [Sov. J. Part. Nucl. **17**, 204 (1986)].
[2]E. Cohen and C. Gomez, Phys. Rev. Lett. **52**, 237 (1984).
[3]G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981).
[4]A. R. Zhitnitskiĭ, Preprint 87-15, Institute of Nuclear Research, Siberian Branch, Academy of Sciences of the USSR; Zh. Eksp. Teor. Fiz. **6**, (1988) [in press].
[5]J. Atick, L. Dixon, P. Griffin, and D. Nemeschansky, Preprint SLAC/PUB-4273/87.
[6]Ya. I. Kogan, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 556 (1987) [JETP Lett. **45**, 556 (1987)].
[7]A. M. Perelomov and M. C. Prafti, Nucl. Phys. **258**, 647 (1985).

Translated by Dave Parsons