Bosonization and calculation of correlation functions in the Wess-Zumino-Witten model

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A representation of $2D$ conformal field theories with a current algebra in terms of free fields is discussed. A Dotsenko-Fateev procedure is proposed. Correlation functions in a $SU(2)_k$ Wess-Zumino-Witten theory are calculated.

1. The hypothesis that the $2D$ conformal field theories which originally arose in a study of statistical models are classical solutions in string theory has recently been adopted widely. On the other hand, the idea that a calculation of correlation functions in any conformal theory can be reduced to a calculation in string theory, i.e., to a representation in terms of free fields, as was done in Ref. 1 for minimal models,[2] has proved to be exceedingly successful. A representation in terms of free fields (a "bosonization") can be particularly useful in a calculation of correlation functions on a surface of a higher kind. Below we propose a Dotsenko-Fateev procedure for calculating correlation functions on a sphere in the simplest Wess-Zumino-Witten model[3] with an $SU(2)_k$ current algebra.[4]

2. The $SU(2)_k$ current algebra can be realized in the following way[1)]:

$$J_+(z)=\frac{i}{\sqrt{2}}w(z),\quad H(z)=iq\partial\phi(z)-w(z)\chi(z)$$
$$J_-(z)=\frac{i}{\sqrt{2}}[w(z)\chi^2(z)-2iq\chi(z)\partial\phi(z)+2(1-q^2)\partial\chi(z)],\tag{1}$$

where w and χ are boson 1- and 0-differentials, ϕ is a scalar field which takes on a value in a circle, $\partial=\partial/\partial z$, and the parameter q is related to the level of the algebra, k, by $2q^2=k+2$. The fields w, χ, ϕ are free in the sense that we can write

$$w(z)\chi(z')=(z-z')^{-1}+\ldots,\quad \phi(z)\phi(z')=-\log(z-z')+\ldots.$$

The system w, χ can be bosonized literally as a boson system of j- and $(1-j)$-differentials[7] (with $j=1$):

$$w=-\partial\xi e^{-u}=-i\partial v e^{-u+iv},\quad \chi=\eta e^{u}=e^{u-iv}$$

$$\xi(z)\eta(z')=(z-z')^{-1}+\ldots,u(z)u(z')=-\log(z-z')+\ldots,v(z)v(z')=-\log(z-z')+\ldots$$

The expressions for the generators in (1) then take the form

$$J_+ = \frac{1}{\sqrt{2}} \partial v e^{-u+iv}, \qquad H = iq\partial\phi + \partial u$$

$$J_- = \frac{1}{\sqrt{2}}[2q\partial\phi - 2iq^2\partial u + (1-2q^2)\partial v]e^{u-iv}. \tag{2}$$

The energy-momentum tensor of the system is determined by a standard Sugawara construction:

$$T = 1/2q^2 : 2J_+J_- + H^2 : = w\partial\chi + T_\phi = T_u + T_v + T_\phi. \tag{3}$$

In other words, the energy-momentum tensor is a sum of "elongated" tensors:

$$T_\varphi = -\frac{1}{2}(\partial\varphi)^2 + i\sqrt{2}\alpha_{0,\varphi}\partial^2\varphi, \quad \varphi = u, v, \phi$$
$$\alpha_{0,u} = i/2\sqrt{2}, \quad \alpha_{0,v} = 1/2\sqrt{2}, \quad \alpha_{0,\phi} = -1/2\sqrt{2}q. \tag{4}$$

Energy-momentum tensor (3), (4) is actually a tensor of a system of *free* scalar fields, in contrast with the case studied in Ref. 8.

3. It is natural to seek conformal fields in a theory with current algebra (2) and energy-momentum tensor (3) as an exponential function of the scalars ϕ, u, and v. Examining the operator expansions with currents (2), we easily verify that the vertex operators

$$V_j = \exp(i\frac{j}{q}\phi), \quad V_{j,-1} = \exp(i\frac{j}{q}\phi)\chi = \exp(i\frac{j}{q}\phi + u - iv),$$

$$V_{j,-2} = \exp(i\frac{j}{q}\phi)\chi^2 = \exp(i\frac{j}{q}\phi + 2(u-iv)), \ldots, V_{j,-2j} \equiv V_{-j}. \tag{5}$$

form an $SU(2)_k$ representation of weight j. The dimensionalities of the operators of series (5) are identical (χ has a zero dimensionality) and are given by

$$\Delta_j = \frac{j(j+1)}{2q^2} = \frac{j(j+1)}{k+2}.$$

The operators of series (5) can be written in the form

$$V_{j,m-j} = \exp(i\frac{j}{q}\phi)\chi^{j-m} = \exp(i\frac{j}{q}\phi + (j-m)(u-iv)),$$

where $m = j, j-1, \ldots, -j$ is the "angular-momentum projection."

4. We turn now to a calculation of correlation functions on a sphere. It follows from the gravitational anomaly[9] that the correlation function in the selected plane metric depends on the point of its singularity or, more precisely, is a differential of degree $-c/3$, where c is the central charge.[10] Since the central charge of the $SU(2)_k$ Wess-Zumino-Witten theory,

$$c = c_\phi + c_u + c_v = (1 - 24\alpha_{0,\phi}^2) + (1 - 24\alpha_{0,u}^2) + (1 - 24\alpha_{0,v}^2) = 3 - 3/q^2 ,$$

differs from that of a system of three free scalar fields, we need to place a "vacuum charge"[1] (with a dimensionality $\Delta_s = 1/q^2$ and a zero angular-momentum projection) at the point of the singularity of the metric, R (at infinity if the metric is $ds^2 = dzdz$):

$$V_s(R) = \exp[\frac{i}{q}\phi(R)]\chi(R) = \exp[\frac{i}{q}\phi(R) + u(R) - iv(R)] . \tag{6}$$

The correlation functions in the theory thus depend explicitly on the point of the singularity of the metric, R, but they can be normalized by some factor, [11] and we can analyze the dependence of only the points at which the operator of the conformal theory are placed.

As a result of this discussion, we can write the following expression for a nonzero two-point function:

$$\langle V_{j,\, m-j}(z)\, \tilde{V}_{j,\, m'-j}(0) \rangle_s \sim \frac{\delta_{m+m',\, 0}}{z^{2\Delta_j}} ,$$

where the average is to be understood as a path integral over the free fields with a *mandatory insertion* of vacuum charge (6). The operators with a tilde (˜) are given by

$$\tilde{V}_{j,\, m-j} \equiv V_{-1-j,\; 1+j+m} \tag{7}$$

They do not form representations of algebra (1), (2), but they do have the same dimensionality and angular-momentum projection as $V_{j,m-j}$.

We turn now to a calculation of the four-point field correlation function in the fundamental representation $j = 1/2$. We use a subscript plus sign for $m = 1/2$, and a minus sign for $m = -1/2$. We have four possible operators:

$$V_+ = \exp(\frac{i}{2q}\phi) , \qquad V_- = \exp(\frac{i}{2q}\phi + u - iv)$$
$$\tilde{V}_+ = \exp(-i\frac{3}{2q}\phi - 2u + 2iv), \quad \tilde{V}_- = \exp(-i\frac{3}{2q}\phi - u + iv) .$$

Let us calculate the correlation function which contains three operators (5) and one operator (7), as in Ref. 1. The correlation function will be of the form

$$\langle \tilde{V}_-(0)\, V_+(x)\, V_+(1)\, V_-(\infty)\, Q \rangle_s , \tag{8}$$

where, instead of insertion (6), we need to insert a so-called Feĭgin-Fuks operator in order satisfy charge conservation.[1,12] A Feĭgin-Fuks operator is an integral of an operator of unit dimensionality over a closed contour. For the theory under consideration here, an operator of unit dimensionality is

$$J(t) = \exp(-\frac{i}{q}\phi(t) - u(t) + iv(t))\,[A\,\partial u(t) + B\,\partial v(t)] .$$

The presence of constants A and B leads to a multiplication of the correlation function by a numerical factor $(A - iB)$, so we can simply set

$$J(t) = \exp(-\frac{i}{q}\phi(t))\,w(t) = -\,i\partial v \exp[-\frac{i}{q}\phi - u + iv] \qquad (9)$$

$$Q = \oint J .$$

Substituting (9) into (8), and going through the elementary calculations, we find

$$\oint dt \langle \tilde{V}_-(0) V_+(x) V_+(1)\, V_-(\infty)\, J(t)\rangle_s$$

$$\propto \oint dt\, t^{(1-k)/(k+2)}(t-1)^{-1/(k+2)}(t-x)^{-1/(k+2)} .$$

Depending on the choice of integration contour, we will have two independent solutions:

$$F(\frac{1}{k+2}, -\frac{1}{k+2}, \frac{k}{k+2}, x); \qquad F(\frac{1}{k+2}, \frac{3}{k+2}, \frac{k+4}{k+2}, x),$$

[$F(\alpha,\beta,\gamma,x)$ is the hypergeometric function] of the Knizhnik-Zamolodchikov equations.[4]

Finally, calculating the field correlation function in this manner,

$$\langle \Phi_{j_1, m_1}(0)\, \Phi_{j_2, m_2}(x)\, \Phi_{j_3, m_3}(1)\, \Phi_{j_4, m_4}(\infty)\rangle, \quad \sum_{i=1}^{4} m_i = 0, \quad j_4 = \sum_{i=1}^{3} j_i - l ,$$

we verify that we need to make precisely l insertions of the operator Q in (9). A simple calculation leads to the following result for the correlation functions:

$$\sum_\gamma C_\gamma \oint \prod_1^l dt_i \prod_{i<j} (t_i - t_j)^{-\frac{1}{q^2} - \gamma_{ij}}$$

$$\times \prod_1^l t_i^{-\frac{j_1}{q^2} - \gamma_i^{(1)}} (t_i - 1)^{-\frac{j^3}{q^2} - \gamma_i^{(3)}} (t_i - x)^{-\frac{j_2}{q^2} - \gamma_i^{(2)}}$$

where

$$\gamma = 0,1: \ \Sigma\gamma = l, \quad \gamma_i^{(1)} = \gamma_i^{(3)} , \qquad (10)$$

and C_γ are certain coefficients. By choosing various integration contours, we can find $l + 1$ independent solutions of the Knizhnik-Zamolodchikov equations in (10) (Ref. 13).

The procedure outlined here can be developed further for the more general case of Wess-Zumino-Witten models.[14]

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[1)]Vl. Dotsenko told the author about this representation (see Ref. 5 and, for the case of an arbitrary algebra, Ref. 6).

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