Fermion chiral determinant in the Schottky parametrization

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An expression for the product of θ functions on Riemann surfaces is derived. This expression can be used to represent a fermion determinant with an arbitrary characteristic as an infinite product in the Schottky parametrization.

1. A key role is played in string theory by the chiral determinants which appear in the expressions for the multiloop string partition functions and amplitudes. These determinants were expressed in terms of θ functions in Ref. 1; in particular, it was shown there that the anomaly-free combination $\lambda_p(E) = (\det \bar{\delta}_0)^{1/2} \det \bar{\delta}_{1/2}(E)$ is equal to $c_p \theta(\alpha/\beta)(\tau)$, where $\det \bar{\delta}_j(E)$ is a chiral determinant in j-differentials on a surface of type p with coefficients from the plane linear stratification E, α_k, and β_k, where $k = 1, p$ are real numbers which specify the splicing functions in this stratifications ($\oint 2$), and the constant c_p depends on only the surface type.

In the approach which was developed in Refs. 3, which makes use of a joining Riemann surfaces with boundaries, the expression for λ_p is "unitary" by construction; i.e., we have $c_p = g^{-2+2p}$, where g is the constant of the three-string interaction. The results, however, are written in the form of an infinite product is the Schottky parametrization. The relationship between the two representations for λ_p follows from the expression derived in this letter for an expansion of a θ function in an infinite product on a Riemann surface. This expression generalized the well-known expansion for type 1 (the Jacobi identity):

$$\theta(z)(\tau) = \prod_{n>0} (1 + \exp(2\pi i z) q^{n-\frac{1}{2}})(1 + \exp(-2\pi i z) q^{n-\frac{1}{2}})(1 - q^n), \qquad (1)$$

where $q = \exp(2\pi i \tau)$.

2. In the Schottky parametrization, a Riemann surface of type p is a Riemann sphere, from which $2p$ disks D_k, D'_k, $k = 1, P$ have been cut out, and the boundaries of the holes—circles S_k, S'_k,—are joined in pairs by fractionally linear transformations $\gamma_k: S'_k = \gamma_k(S_k)$. The transformations γ_k generate the group G, which is called the "Schottky group." An arbitrary of G can be written in the form

$$\gamma = \gamma_{i_1}^{m_1} \cdot \ldots \cdot \gamma_{i_N}^{m_N} .$$

We define $n_k(\gamma)$—the number of entries of generators in γ—by $n_k = \Sigma_{\alpha, i_\alpha = k} m_\alpha$.

Each element of Schottky group G is a dilatation in some coordinate system which is found from the original coordinate system by means of a fractionally linear transformation ψ:

$$\psi_0 \gamma_0 \psi^{-1}(z) = K_\gamma z .$$

The number K_γ is the "factor" of element γ.

As the A_k cycles on the joined Riemann surface Σ_p we select the circles S_k, while as the B_k cycles we select certain curves which connect points $P_k \in S_k$ and $\gamma_k(P_k) \in S'_k$.

A stratification on Σ_p is specified by the splicing functions on the A and B cycles; for example, a stratification of j-differentials corresponds to splicing functions which are equal to 1 on B cycles and $(\gamma'_k(z))^j$ on A_k cycles.

Among the 2^{2p} spinor structures there is a basis structure which is associated with the choice of A and B cycles described above. Its splicing functions are equal to 1 on B cycles and $\sqrt{\gamma'_k(z)}$ on A_k cycles; the choice of the sign of the root is matched in the following way with the choice of B cycles: We fix a circle S_k and a point P_k on it. We consider a path $\gamma_k(z,t)$, $0 \leqslant t \leqslant 1$, in the group of fractionally linear transformation which connects unity and $\gamma_k(z)$ and which is of such a nature that the point $P_k(t) = \gamma_k(P_k, t)$ moves along a B_k cycle from P_k to $\gamma_k(P_k)$. The sign of the root $\sqrt{\gamma'_{k'}(z)}$ is then reconstructed by continuity $\sqrt{\gamma'_k(z,0)} = \sqrt{1} = 1$). The choice of the sign of $\sqrt{\gamma'_k(z)}$, $k = 1, p$ determines the sign of $\sqrt{\gamma'(z)}$ for any γ from the Schottky group and makes it possible to determine the correct sign of the $K_\gamma^{1/2}$. The 2^{2p} spinor structures on Σ_p are specified by two p-dimensional integer vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ structure corresponds to splicing functions which are equal to $\sqrt{\gamma'_k(z)}\,(-1)\ B_k$ on A_k and $(-1)^{\alpha_k}$ on B_k.

The stratification of j-differentials, which has been multiplied in a tensor fashion by an arbitrary plane stratification E corresponds to the splicing functions described above with real vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

3. The basic result of this letter is a representation of a θ function as an infinite product on a Riemann surface:

$$\theta(\mathbf{Y})(\hat{\tau}_p) = \prod_\gamma{}' \prod_{m=1}^{\infty} (1 + \exp(2\pi i \mathbf{Y}\cdot\mathbf{n}) K_\gamma^{m-\frac{1}{2}})$$

$$\times (1 + \exp(-2\pi i \mathbf{Y}\cdot\mathbf{n}) K_\gamma^{m-\frac{1}{2}})(1 - K_\gamma^m), \qquad (2)$$

where the product is calculated from the primitive classes of conjugate elements (an element $\gamma \in G$ is primitive if $|K_\gamma| < 1$ and $\gamma = \tilde{\gamma}^q$; for $\tilde{\gamma} \in G$, it is primitive only for $q = \pm 1$), $\hat{\tau}_p$ is the matrix of periods of surface $\Sigma_{p'}$ calculated for the choice of A and B cycles described above, and $\mathbf{n} = \mathbf{n}(\gamma)$.

The idea underlying proof (2) is that for real $\mathbf{Y}$ the left side of (2) is equal to $\lambda_p(E)$ for a stratification of E with $\boldsymbol{\alpha} = 0$, $\boldsymbol{\beta} = 2\mathbf{Y}$ (Ref. 1), and in this case expression (2) can be proved by integrating the energy-momentum tensor over moduli in the Schottky representation (this was done by Martinec[2] for $\mathbf{Y} = 0$). The validity of (2) then follows from analyticity in terms of the left and right sides of relation (2).

We can prove (2) for real $\mathbf{Y}$:

$$\det \bar{\partial}_{1/2}(E) = \int D\psi\, D\tilde{\psi}\, \exp\left(\int \tilde{\psi}\, \dot{\bar{\partial}}\, \psi\right), \qquad (3)$$

where $\psi(\tilde{\psi})$ is a $(1/2)$-differential with coefficients from a plane stratification of $E(E^{-1})$. The holomorphic energy-momentum tensor of a system of this sort is $T=(1/2)(\tilde{\psi}\partial\psi-(\partial\tilde{\psi})\psi)$. Following Martinec, we find the vacuum expectation value of this tensor from a Green's function:

$$\langle T\rangle = \lim_{z\to w}\left[\frac{1}{2}\partial_w G(z,w)-\frac{1}{2}\partial_z \tilde{G}(z,w)-\frac{1}{(z-w)^2}\right], \qquad (4)$$

where

$$G(z,w)=\langle\tilde{\psi}(z)\psi(w)\rangle .$$

The Green's function $G(z,w)$ has a single second-order pole and is determined unambiguously by its automorphic properties:

$$\begin{aligned} G(z,\gamma_k(w)) &= \exp(2\pi i Y_k)\sqrt{\gamma_k'(w)}\,G(z,w), \\ G(\gamma_k(z),w) &= \exp(-2\pi i Y_k)\sqrt{\gamma_k'(z)}\,G(z,w). \end{aligned} \qquad (5)$$

Hence

$$G(z,w)=\sum_{\gamma\in G}\frac{\sqrt{\gamma'(z)}\exp(-2\pi i\mathbf{Y}\cdot\mathbf{n})}{\gamma(z)-w},$$

where the sum is over all elements of the Schottky group. Pursuing the analysis by Martinec's approach, and using his result for $(\det\partial_0)^{1/2}$, we find that $\lambda_p(E)$ for real $\mathbf{Y}$ is equal to the right side of (2) within a constant which depends on only the type. The constant in relation (2) is established when we make the transition to a degenerate surface, $K_{\gamma_k}\to 0$. This completes our proof of expression (2) for real $\mathbf{Y}$.

4. According to Knizhnik,[1] we have

$$\lambda_p(E)=c_p\exp 1/4\,(i\pi\vec{\alpha}\hat{\tau}_p\vec{\alpha}+2\pi i\vec{\alpha}\cdot\vec{\beta})\theta(1/2(\vec{\beta}+\hat{\tau}_p\vec{\alpha}))(\hat{\tau}_p), \qquad (6)$$

The substitution of infinite product (2) into (6) gives us a representation for $\lambda_p(E)$ in the Schottky parameterization. We will derive it by a joining method in a following paper.

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