Ultrastrong wave collapse

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A wave collapse described by a nonlinear Schrödinger equation can lead to the formation of a "hot spot" which absorbs energy from the surrounding space. Various regimes of this "ultrastrong" collapse are analyzed.

1. In the theory of wave collapses there have already been suggestions[1–3] that a collapse might result in the formation of a small, long-lived dissipation zone (a "hot spot"), localized in all dimensions, which absorbs wave energy from the surrounding space. This effect has been termed a "funnel effect,"[1] a "nucleation,"[2] and a "distributed collapse."[3] In the present letter we wish to suggest the name "ultrastrong collapse" for this effect, which is extremely important from a variety of points of view. We will discuss the appearance of ultrastrong collapse in the most fundamental model of wave collapse: the nonlinear Schrödinger equation with the maximal spatial symmetry,

$$i\psi_t + \psi_{rr} + \frac{d-1}{r}\psi_r + |\psi|^s\psi = 0. \tag{1}$$

We are leaving the dimensionality of the space, d, arbitrary (it may be fractional).

According to Eq. (1), a collapse occurs if $sd \geqslant 4$. At $sd = 4$ (the "critical case"[4–7]) the collapse is strong: A fixed amount of energy goes into the collapse point, a finite part of this energy is dissipated, and then the collapse regime gives way to an expansion regime. At $sd > 4$ ("supercritical collapse") the collapse point is approached in accordance with the self-similar law

$$\psi(r,t) = (t_0 - t)^{-1/s - i\kappa} g\left(\frac{r}{\sqrt{t_0 - t}}\right). \tag{2}$$

The validity of this assertion has been verified by several numerical calculations.[6–8] If an expansion occurs after the collapse in this case, the collapse is weak: The energy dissipated in the collapse zone vanishes with a decrease in the nonlinear damping coefficient. Actually, however, the collapse is weak only under the inequality

$$4 < sd < 2s + 2, \quad s > 1. \tag{3}$$

In the opposite case, $sd \geqslant 2s + 2$ (which holds only for $d > 2$), the development of the collapse may result in an ultrastrong collapse: the formation of a hot spot which absorbs energy in a quasisteady fashion.

2. This possibility arises because at $sd > 2s + 2$ Eq (1) has the exact singular solution

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$$\psi = \frac{A}{r^{2/s}}, \qquad A = [\frac{2}{s^2}(sd - 2s - 2)]^{1/s}. \tag{4}$$

For $2s + 2 < sd < 2s + 4$, solution (4) is supplemented by a family of steady-state singular solutions which have the following asymptotic behavior as $r \to 0$:

$$|\psi| = \frac{A}{r^{2/s}}(1 + A_1 r^{\mu} + \ldots), \qquad \mu = \frac{2}{s}(2s + 4 - sd) > 0, \qquad A_1 = qP^2, \tag{5}$$

where $P = -\lim_{r\to 0} |\psi|^2 r^{d-1} (d/dr) \mathrm{Arg}\, \psi$ is the energy flux to the singularity [see also Ref. 9; we are not reproducing the expression for the constant $q = q(s,d) > 0$]. At $d = 3$ this case prevails at nonlinearity indices $2 < s < 4$. In the case $sd = 2s + 4$ ($s = 4$ if $d = 3$), Eq. (1) has a single-parameter family of steady-state singular solutions, for which we have

$$|\psi| = \frac{B}{r^{2/s}}, \qquad B^4 [B^s - (\frac{2}{s})^2] = P^2 . \tag{6}$$

If $sd > 2s + 4$, solution (4) is isolated, but Eq. (1) has a single-parameter family of "quasiclassical" (in the sense of Ref. 10) solutions with the asymptotic behavior

$$|\psi| = \frac{c}{r^{\gamma}}(1 + C_1 r^{\nu} + \ldots), \tag{7}$$

where $C = P^{\alpha}$, $\gamma = \alpha(d-1)$, $v = \alpha(sd - 2s - 4) > 0$, $\alpha = 2/s + 4$, and $C_1(s,d) > 0$.

Finally, at the lower limit of the parameter region under consideration, $sd = 2s + 2$ (the case $d = 3$, $s = 2$, which is a particular version of this limit, but physically the most important one, was analyzed in Refs. 3 and 9), the following special steady-state solution can hold:

$$|\psi| = \frac{(2/s^2)^{1/s}}{r^{2/s}\, |\ln r|^{1/s}} . \tag{8}$$

Near this solution there is also a family of singular solutions with a flux. The existence of steady-state singular solutions of Eq. (1) with a nonzero flux is analogous to the case of a "falling on a center" in quantum mechanics (Ref. 11, for example). Since the absorption of energy upon the formation of such solutions may be significantly greater (over a fairly long time) than the absorption of energy in a single strong collapse, these regimes could reasonably be called "ultrastrong collapses."

3. To test the fact that steady-state regimes of ultrastrong collapse are established, we carried out a numerical integration of Eq. (1), supplemented with a term of the form $i\beta|\psi|^m\psi$, which corresponds to the introduction of a nonlinear damping lumped near the singularity at $r \approx 0$. The calculations were carried out in terms of the variables $d\tau/dt = |\psi(0,\tau)|^s$, $\xi = r|\psi(0,\tau)|^{s/2}$ by the method of Refs. 5 and 7. Below we report results for $s = 2$, $\beta = 10^{-9}$, $m = 6$, and $\psi(r,0) = \exp(-r^2/16)$. We varied the dimensionality d over the range $2.5 \leqslant d \leqslant 5$ in such a way that the grid of versions covered

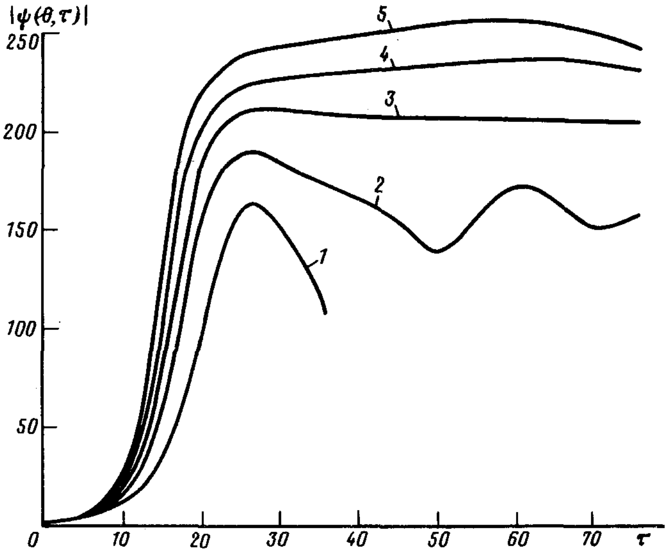

FIG. 1. Evolution of the field at the center for $s=2$ and various versions. 1—$d=2.5$; 2—$d=3$; 3—$d=3.5$; 4—$d=4$; 5—$d=5$.

all possible dynamic regimes. When we fixed the dimensionality at $d=3$ and varied s, we found no qualitative change in the picture.

In all versions we observed that the solution reliably approached the mode of a weak self-similar collapse in the inertial interval, to the point that a nonlinear damping came into play at the level $|\psi(0,\tau)|^2 \sim 10^4$–$10^5$. We see from Fig. 1, which shows the time evolution of the amplitude at the center, that the behavior of the solution at $d=2.5$ is characteristic of the ordinary scenario of weak collapse. After the energy has been depleted, an expansion begins. At $d=3$ we observe an oscillatory regime, which we can somewhat arbitrarily regard as a "quasisteady" regime. For the versions $d=3.5$, $d=4$, and $d=5$ the behavior of the amplitude is approximately in a steady state. The hypothesis of the existence of hot spots is thus effectively confirmed. We also found an agreement between the spatial behavior of the collapse mode which was established and the equations given above. For example, we see from Fig. 2 that the quantity $F(\xi)=\xi^{\gamma}|\psi(\xi,\tau)|/|\psi(0,\tau)|^{\gamma s/2}$ (for the versions shown here, we are using the value $\gamma=2/s=1$) becomes essentially independent of the spatial variable outside the effective radius of the nonlinear damping. Finally, the existence of steady-state singular solutions (5)–(7) of Eq. (1) was verified by our direct numerical integration of the steady-state equation with a discontinuous damping coefficient $\beta=\beta_0\theta(r_0-r)$, $r_0 \ll 1$.

We wish to stress that hot spots which are supplied by a constant energy flux can exist regardless of the degree of the nonlinearity only in a case with $d>2$ (realistically, only for three-dimensional physical systems). In cases with $d \leqslant 2$ there can be (at $sd>4$) only a weak collapse, which is what was observed in Ref. 8 in an analysis of Eq.

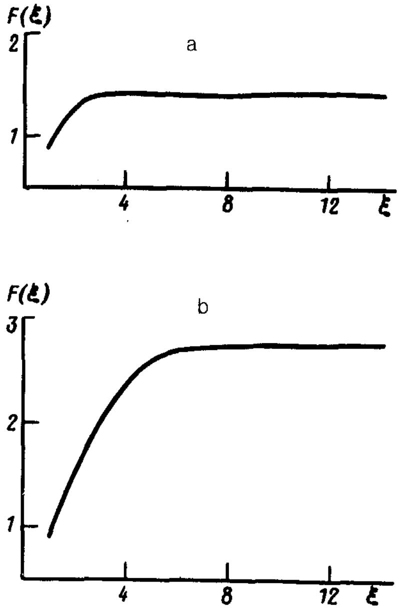

FIG. 2. Spatial structure of a solution in the quasisteady state for two versions: a—$d = 3.5$; b—$d = 4$.

(1), with $d = 1$ and $s = 6$, as a model suggested for the three-dimensional problem. It is now clear that this assumption is unjustified and that constancy of the product sd does not guarantee a qualitative similarity in the behavior of the solutions as the dimensionality is reduced.

We are understanding the "quasisteady nature" of the regime of ultrastrong collapse in the sense that the lifetime of the regime is significantly longer than the time scale of the preceding evolution in the inertial interval.

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