Quartions in relativistic field theory

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The equations of motion and the Lagrangian for massless particles with a helicity $\pm 1/4$ (quartions) are analyzed in spaces of dimensionalities $D=(2+1)$ and $D=(3+1)$.

The possible values of the spin and statistics of solitons in relativistic quantum-mechanical models with $D=(2+1)$ have attracted considerable research interest in recent years. It has been shown[1] that if the Lagrangian of such a model contains certain topological terms, the spins and statistics of the solitons can be fractional.[1] For the case of the $SU(2)$ gauge symmetry group, the spin of a soliton can take on the values $s=1/4+m$ and $3/4+m$ ($m \in Z$) (Ref. 2).

There is accordingly the problem of constructing a relativistic quantum-mechanical model with $D=(2+1)$ which would directly describe local entities with these spin values. One might expect that the relationship between such a model and the $D=(2+1)$ σ models, in which there are corresponding solitons, would be analogous to that between the $D=(1+1)$ Thirring and sine-Gordon models. Even more interesting would be the derivation of a systematic quantum theory of quartions[1)] in a space with $D=(3+1)$. Such a theory might be directly pertinent to elementary particle physics.

In the present letter we discuss the possibility of describing, at the level of classical fields, quartions with a zero mass in spaces with $D=(2+1)$ and $D=(3+1)$ through the use of infinite-dimensional representations of the $SL(2,R)$ and $SL(2,C)$ groups with weights of 1/4 and 3/4.

How can quartions in the space with $D=(3+1)$ be described? The representations $(-1/4,0)$, $(-3/4,0)$ and $(0,-1/4)$, $(0,-3/4)$ of the $SL(2,C)$ group, which correspond to these spin values, could be constructed by introducing spinor operators with commutation relations

$$[L_{\dot\alpha}, L_{\dot\beta}] = -i\epsilon_{\dot\alpha\dot\beta}; \qquad [L_\alpha, L_\beta] = i\epsilon_{\alpha\beta}. \tag{1}$$

The generators of the transformations of the $SL(2,C)$ group are written as the anticommutators of the operators $L_{\dot\alpha}$ and L_α (Refs. 3 and 4):

$$M_{\dot\alpha\dot\beta} = \frac{1}{4}\{L_{\dot\alpha}, L_{\dot\beta}\}, \qquad M_{\alpha\beta} = \frac{1}{4}\{L_\alpha, L_\beta\}. \tag{2}$$

The equations which we are proposing for the quartions are

$$L_{\dot\alpha}\nabla^{\dot\alpha\beta}\bar\phi = 0, \tag{3a}$$
$$L_\alpha\nabla^{\alpha\dot\beta}\phi = 0. \tag{3b}$$

It follows from (3) that we have $\nabla^2\bar{\phi} = \nabla^2\phi = 0$, i.e., that Eqs.(3) describe a particle of zero mass.

Applying a Pauli-Lyubanskiĭ vector $S_a = (1/2)\epsilon_{abcd}M^{bc}P^d$ to functions $\bar{\phi}$ and ϕ which satisfy Eqs. (3), and using Eqs. (3), we find $S_a = \pm(1/4)P_a$. In other words, the helicities of the states $\bar{\phi}$ and ϕ are $+1/4$ and $-1/4$, respectively, and these states are superpartners of each other, forming the simplest possible supermultiplet.[2)] Equations (3) do not permit the incorporation of the interaction with gauge fields. The reason is that Eqs. (3a) and (3b) each constitute a system of two equations ($\beta,\dot{\beta} = 1,2$), which ceases to be compatible when an interaction is turned on.

To find a modification of Eqs. (3) which would not present this difficulty, we rewrite Eq. (3a) for a plane wave:

$$\bar{\phi} = \bar{\phi}(\lambda)\exp(i\pi_\alpha\pi_{\dot{\alpha}}x^{\alpha\dot{\alpha}}) \tag{4}$$

(λ is the index of the basis vectors of the space of the operators L_α). We then find from (3a)

$$L^{\dot{\alpha}}\pi_{\dot{\alpha}}\pi_\alpha\bar{\phi}(\lambda) = 0 \tag{5}$$

or, equivalently,

$$L^{\dot{\alpha}}\pi_{\dot{\alpha}}\bar{\phi}(\lambda) = 0. \tag{6}$$

Although the transformation from Eq. (5) to Eq. (6) is trivial, Eq. (6) makes it possible to analyze the scattering of a quartion by the field of an electromagnetic or gravitational wave by the methods of twistor theory.[6] The structure of Eqs. (5) and (6) suggests the following form of the action integral for the interacting quartions:

$$S = \int d\lambda d\bar{\lambda} d^4x\, \{(\phi C L_\alpha \phi)(\lambda, x)[\bar{\phi}\bar{C}\bar{L}_{\dot{\alpha}}\nabla^{\dot{\alpha}\alpha}\bar{\phi} - (\nabla^{\alpha\dot{\alpha}}\bar{\phi})\,\bar{C}L_{\dot{\alpha}}\bar{\phi}]\,(\bar{\lambda}, x)$$
$$+ \text{the complex conjugate term}\}, \tag{7}$$

where the operator C is found from the condition $CL_\alpha C^{-1} = iL_\alpha^T$. In the coordinate representation we would have $C = \exp(-(1+i)\lambda(\partial/\partial\lambda))$ (where the tilde $\tilde{}$ means the right-hand ordering of the derviative $\partial/\partial\lambda$ with respect to the variable λ). It is easy to verify that we have $C^T + C^{-1}$ and $C^4 = 1$.

Since the functions ϕ are defined by means of four-valued representations of a Lorentz group, the quantization of action (7) should be carried out in accordance with fractional statistics.

In general, the function ϕ in the representation of $(-1/4, 0)$ and $(-3/4, 0)$ depends not only on the variable λ but also on the variable $\bar{\lambda}$. Action (7) can be generalized to take this dependence into account quite easily, by the standard Clebsch-Gordan technique.

Lagrange's equations for (7) contain solutions in the form of plane waves, (4); in this case the product $\pi_\alpha\pi_{\dot{\alpha}}$, which determines the 4-momentum of a quartion, is factorized in a symmetric fashion with respect to factors with L_α and $L_{\dot{\alpha}}$ in (7). The nonlinearity in (7) leads to a rescattering of a quartion by a quartion. An interaction with gauge fields could be introduced in (7) in the standard way.

The analysis of relativistic theories with $D=(2+1)$ is carried out in a similar way. The operators L_α in (1) and (2) are determined in this case by indices α of a common type, while the operators $M_{\alpha\beta}$ in (2) are the generators of $SL(2,R)$ transformations. The bilinear form $\int(\phi_2 C\phi_1)(\lambda)d\lambda$, which is positive definite under the condition $\phi_2^+ = \phi_2 C$, can be used to construct invariants. As individual terms in the action, on the same level as (7), we could use a bilinear kinetic term or a bilinear mass term. At this point it is not clear just which versions of the actions are systematic theories, and just how systematic, when they are quantized in accordance with a fractional statistics.

In conclusion we will use the example of action (7) for the case $D=(2+1)$ to analyze the possibility of modifying relativistic field theories in which the bosons and fermions can be interpreted as composite entities consisting of quartions. We replace the minus sign by a plus sign in square brackets in action (7). As a result, the expression inside the brackets becomes a total derivative, and action (7) becomes the action of a free neutrino. For arbitrary constants in front of the terms in brackets, action (7) contains both a kinetic term for a composite neutrino and terms which determine its interaction with a quartion.

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[1)] A Russian translation of "quartion" is *chetvertinka.*

[2)] Yu. P. Stepanovskiĭ has called our attention to the circumstances that Eqs. (3) are the limiting case of an infinite-component Dirac equation[5] with $m=0$. The wave function which was used in Ref. 5, however, contained representations $(-1/4, -1/4)$, etc., in contrast with (3). As a consequence, the helicity of massless Dirac particles is zero.

[1] F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983).

[2] A. Balachandran, M. Bowick, K. Gupta, and A. Srivastava, Mod. Phys. Lett. A **3**, 1725 (1988).

[3] S. Sannikov, Ukr. Fiz. Zh. **10**, 684 (1965); S. S. Sannikov, Teor. Mat. Fiz. **34**, 34 (1978).

[4] A. M. Penelomov, *Generalized Coherent States and Their Applications*, Nauka, Moscow, 1987.

[5] P. Dirac, Proc. R. Soc. A **322**, 435 (1971); **328**, 1 (1972).

[6] R. Penrose and M. McCallum, Phys. Rep. **6**, 241 (1972).

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