Soliton diffusion coefficient

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A soliton gas in systems which are nearly exactly integrable is characterized by two fundamentally different diffusion coefficients. The coefficient D_* describes a process in which the coordinate of the soliton becomes stochastic against the background of a motion at a constant velocity. The coefficient D, in contrast, is related to the viscosity of the soliton, η, by the Einstein relation. The value of this coefficient, in contrast with D_*, is nonzero only in systems which have a disrupted integrability. It is important to take both of these coefficients into account in describing experiments.

Research on the central peak in the susceptibility $\chi(q,\omega)$ by the method of inelastic neutron scattering has demonstrated the reality of soliton excitations of quasi-1D ordered media (ferromagnets, antiferromagnets, ferroelectrics, etc.[1,2]). The first theoretical papers,[3,4] which used the approximation of a free motion of solitons, predicted a Gaussian shape for this central peak: $\chi(q,\omega) \sim \exp\{-m_*\omega^2/2Tq^2\}$. This prediction contradicts the experiments of Ref. 5, which revealed a Lorentzian shape. Such a shape is characteristic of a stochastic (diffusive) motion of particles,[6] and one is led to ask whether the motion of the soliton becomes stochastic as a result of an interaction with a reservoir of quasilinear perturbations (for definiteness, magnons). The first calculation at an elementary level was carried out in Refs. 7 and 8, for a φ^4 model. That calculation predicted a diffusion coefficient D proportional to $\overline{W}$, which is the average probability for the scattering of magnons by a soliton. According to Refs. 7 and 8, the diffusion coefficient obeys $D \sim T^2$. Corresponding results ($D \sim \overline{W}$, $D \sim T^2$) have been derived for an exactly integrable sine-Gordon model.[9]

The results of Refs. 7–9 came under criticism in Refs. 10–13, on the basis of the following arguments. 1. The Einstein relation tells us that we have $D = T/\eta$, where η is the soliton viscosity coefficient which results from soliton-magnon collisions, so we have $D \sim 1/\overline{W}$ and $D \to \infty$ as $T \to 0$. 2. In an exactly integrable system of the sine-Gordon type, there is no irreversibility; in particular, we have $\eta = 0$ and $D = \infty$. These arguments are convincing, but theories of the types in Refs. 7–9 give good descriptions of the experimental data.[5] Consequently, the diffusion of solitons and the shape of the central peak in the susceptibility remain open questions.

Let us consider a soliton with a coordinate $x(t)$ which is interacting with magnons. We denote by α_k and ω_k the magnon amplitudes and frequencies (we are restricting the calculation to the classical case), where k is the momentum of the magnon (everywhere, we are setting $\hbar = 1$). Equations for x and α_k can be written in the form

$$m_* \ddot{x} = \delta H_{int}/\delta x, \qquad i\dot{\alpha}_k = \omega_k \alpha_k + \delta H_{int}/\delta \alpha_k^*, \tag{1}$$

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where m_* is the mass of the soliton, $\dot{f} \equiv df/dt$, and $H_{\rm int}$ is the Hamiltonian (more precisely, Routh function) which describes the interaction of the soliton with the magnons. For a wide range of systems which are nearly integrable [the sine-Gordon equation,[10,13] the φ^4 system (Ref. 13), a ferromagnet,[14] etc.], $H_{\rm int}$ can be written

$$H_{int} = \sum_{12} (\dot{x}T_{12} + \epsilon U_{12}) e^{i(k_2 - k_1)x(t)} \alpha_1^* \alpha_2 + \dots . \tag{2}$$

Here $k_1 \equiv 1$; we have an amplitude $T_{12} \neq 0$ even for exactly integrable systems; and the parameter $\epsilon \ll 1$ determines the rate at which the exact integrability is disrupted (for example, in a transition from the sine-Gordon model to the double sine-Gordon model, of the form $\ddot{\varphi} - \varphi'' + \sin\varphi + \epsilon \sin 2\varphi = 0$). The main property of the amplitudes is $T_{12} = 0$ at $\omega_1 = \omega_2$, and in this case we have $U_{12} \neq 0$ (Refs. 10, 13, and 14). In writing (2) we omitted terms of the type $\alpha_1\alpha_2$, $\alpha_1^*\alpha_2\alpha_3$, etc., and also terms containing $\dot{x}^2$ and $\dot{x}^3$ (we are assuming that the velocity of the soliton is low).

By virtue of (1) we have $m_*\ddot{x} = f(x,\{\alpha_k\})$, $f = -\delta H_{\rm int}/\delta x$; i.e., the force acting on a soliton is a functional of α_k. Working in a perturbation theory in a $H_{\rm int}$ (i.e., in $\dot{x}$ and ϵ), we can write an explicit expression for $f(t)$. In the leading approximation we have $\alpha_k(t) = c_k \exp(-i\omega_k t)$, and $f(t)$ is a random force with $\langle f(t) \rangle = 0$ [when we take an average over the reservoir of magnons, we use the customary rules $\langle c_k \rangle = 0$, $\langle c_1^* c_2 \rangle = (T/\omega_1)\delta_{12}$. Here T is the temperature (we are assuming $\omega_k \ll T \ll E_0$, where E_0 is the energy of the soliton). For the correlation function of the random force f we find, in the leading approximation in x and ε,

$$\langle f(t) f(0) \rangle_0 \approx 2T\eta\delta(t) + 2D_* m_*^2 (-\ddot{\delta}(t)), \tag{3}$$

where

$$\eta = \pi T \epsilon^2 \sum_{12} (k_1 - k_2)^2 |U_{12}/\omega_1|^2 \delta(\omega_1 - \omega_2), \tag{4}$$

$$D_* = (\pi T^2/m_*^2) \sum_{12} |T_{12}/\omega_1|^2 \delta(\omega_1 - \omega_2). \tag{5}$$

The coefficient η in (3) has the meaning of the magnetic viscosity of a kink, and we have the value $\eta = 0$ at $\epsilon = 0$ (this result is actually a consequence of the property $T_{12} = 0$ at $\omega_1 = \omega_2$, i.e., a result of the reflectionless nature of the collisions of a magnon and a soliton in exactly integrable systems). The contribution U_{12} describes the momentum transfer in the course of collisions and leads to both viscosity [in the next higher order of the perturbation theory in ϵ we have $\langle f(t) \rangle = -\eta\dot{x} \neq 0$] and an ordinary diffusion, with coefficient $D = T/\eta$. This diffusion determines a Brownian motion of the soliton. The coefficient D_* in the term with the second derivative of the δ-function in the correlation function, on the other hand, describes an effect which we will call "D_* diffusion." The reason for this effect is that even in an exactly integrable system a soliton undergoes a displacement along the coordinate as it interacts with a magnon. Since collisions occur at random times, the effect is again to make the motion of the soliton stochastic, but against the background of a motion with a constant velocity. The second term in (3) of course does not contribute to the viscosity; the form of D_* corresponds to the results of Refs. 7–9.

The dynamics of the soliton can thus be described by the equation

$$m_* \ddot{x} + \eta \dot{x} = f(t), \quad (6)$$

where f is a random process (for simplicity, we assume it to be Gaussian) with correlation function (3). If only the D diffusion ($D_* = 0$) or only the D_* diffusion ($\eta = 0$) is taken into consideration, we easily find from (6)

$$\langle (x(t) - x(0))^2 \rangle_D \to 2Dt \quad \text{at} \quad t \gg m_*/\eta; \ \langle (x(t) - x(0) - \dot{x}(0)t)^2 \rangle_{D_*} = 2D_* t. \quad (7)$$

If both types of interactions are taken into consideration at early times ($t < \tau_r = m_*/\eta$, where τ_r is the viscous relaxation time), the D_* diffusion "operates"; at late times ($t > \tau_r$), the soliton executes an ordinary Brownian motion.

Let us find the shape of the central peak determined by the soliton component of the imaginary part of the susceptibility, $\chi''(q,\omega)$. We know[15,16] that $\chi''(q,\omega)$ is proportional to the integral $I(q,\omega) = 2\int_0^\infty dt \cos \omega t \exp[-(q^2/2)\langle \Delta x^2(t) \rangle]$, where $\Delta x(t) = x(t) - x(0)$. From this formula we find, in the limit $D_* \gg D$,

$$I(q,\omega) \approx \frac{2D_* q^2}{\omega^2 + (D_* q^2)^2}(1 - e^{-D_* q^2 \tau_r}) + \frac{2Dq^2}{\omega^2 + (Dq^2)^2} e^{-3D_* q^2 \tau_r/2}. \quad (8)$$

In the (more realistic) opposite case $D_* \ll D$, the asymptotic behavior is more complicated, but for extremely small and large values of q it is qualitatively the same as (8). Under the conditions $q < 1/l_c$ ($l_c = \sqrt{D\tau_r}$ is the mean free path) and $q > 1/l_c^* \equiv \sqrt{T/m_* D_*^2}$, Lorentzian peaks form with diffusion coefficients D and D_*, respectively. In the intermediate region, the shape of the central peak is reminiscent of a Gaussian peak, which would be characteristic of free motion. Estimates of the characteristic values of $(1/l_c)$ and $(1/l_c^*)$ for a sine-Gordon model with the standard parameter values[2] and $\epsilon = 0.1$ and $T \sim E_0$ yield 10^{+5} cm^{-1} and 10^{+7} cm^{-1}, respectively. Consequently, the contributions of the D diffusion and D_* diffusion can be distinguished in a neuton experiment if the scattering angle is chosen appropriately. The contribution of D diffusion can also be found from the absorption of sound or electromagnetic waves with $q < 1/l_c$.

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