New corrections to hyperfine splitting in muonium and hydrogen

S. G. Karshenboĭm, V. A. Shelyuto, and M. I. Éĭdes
D. I. Mendeleev All-Union Scientific-Research Institute of Metrology

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Radiative corrections to the hyperfine splitting of relative order $\alpha^2(Z\alpha)$, induced by polarization insertions in external photons, are derived. The new contribution is $(\alpha^2(Z\alpha)/\pi)(-4/3\ln^2[(1+\sqrt{5})/2] - 20/9\sqrt{5}\ln[(1+\sqrt{5})/2] + 608/45\ln 2 + (\pi^2/9) - (38)/15\pi + 91639/37800)$ in units of the energy of the Fermi hyperfine splitting, E_F.

After the radiative corrections to the recoil are calculated for the hyperfine splitting in the ground state of muonium,[1-3] only purely radiative corrections of order $\alpha^2(Z\alpha)E_F$ could lead to contributions at the level of a few kilohertz. These corrections are about an order of magnitude greater than the error of the experiment of Ref. 4, which was 0.16 kHz. Determining these corrections is thus an extremely urgent matter.

Using the methods developed in Ref. 5, one can easily show that the matrix elements of the six gauge-invariant sets of diagrams in Fig. 1 exhaust the total number of corrections of order $\alpha^2(Z\alpha)E_F$. The hatched area in part c of Fig. 1 represents all of the renormalized insertions of one radiative photon in an electron line. These insertions dress a two-photon emission. The hatched area in part f represent all insertions of two radiative photons. We can now calculate the contributions to the hyperfine splitting from diagrams a-c, which contain insertions of a polarization operator in external photons.

Corresponding to the skeletal diagram with two external photons is the infrared-divergent integral[2]

$$\frac{8Z\alpha}{\pi} E_F \int_0^\infty \frac{dk}{k^2}, \tag{1}$$

where $k = |\mathbf{k}|$ is the 3-momentum of the exchange photon, which has been rendered dimensionless by means of the mass of an electron. In contrast with Refs. 2 and 3, we have included the total magnetic moment of a muon, rather than the Dirac magnetic moment of the muon, as a factor in the definition of the Fermi energy E_F. Our reason is that all of the corrections found below stem from integration momenta which are small in comparison with the mass of a muon. At these momenta, the anomalous magnetic moment enters all physical quantities on an equal basis with the Dirac moment according to the low-energy theorem.

The insertion of two single-loop polarization operators in an external photon in

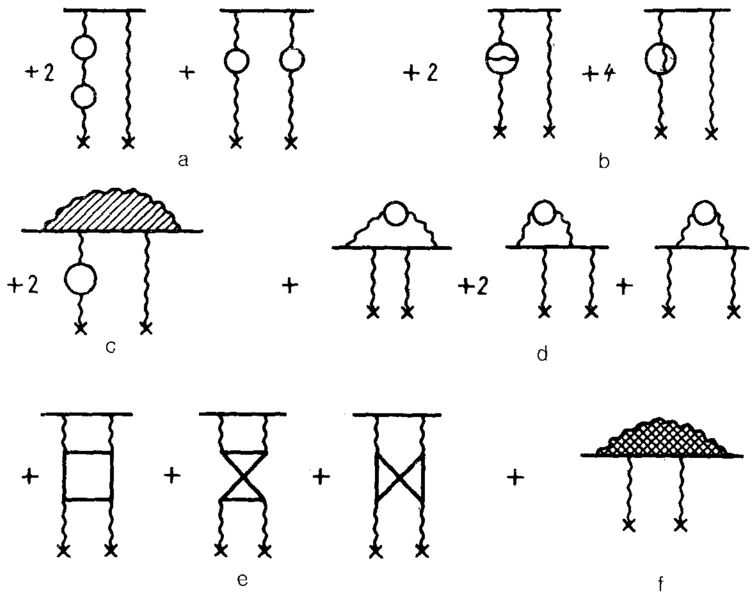

FIG. 1. Gauge-invariant sets of diagrams which lead to contributions of order $\alpha^2(Z\alpha)E_F$ to the hyperfine splitting.

part a of Fig. 1 corresponds to the following replacement in the integrand in (1):

$$\frac{1}{k^2} \to k^2\left(\frac{\alpha}{\pi}\right)^2 \left[\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}\right]^2. \tag{2}$$

The corresponding contribution to the hyperfine splitting is described by the integral

$$\delta E_{1P} = \frac{3\times 8\alpha^2(Z\alpha)}{\pi^3} E_F \int_0^\infty dk k^2 \left[\int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}\right]^2, \tag{3}$$

where the combinatorial factor of 3 incorporates the existence of three diagrams in Fig. 1a. Evaluating first the intregral over v and then that over k, we find

$$\delta E_{1P} = \frac{36}{35}\times \frac{\alpha^2(Z\alpha)}{\pi} E_F. \tag{4}$$

The diagrams in Fig. 1b are taken into account by means of the substitution

$$\frac{1}{k^2} \to \frac{2}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{dv \cdot v}{4+k^2(1-v^2)} \times \Big\{ \frac{1}{2}(3-v^2)(1+v^2)\Big[\frac{\pi^2}{6} + \ln\frac{1+v}{2}\ln\frac{1+v}{1-v}$$

$$+ 2Li\left(\frac{1-v}{1+v}\right)$$

$$+ 2Li\left(\frac{1+v}{2}\right) - 2Li\left(\frac{1-v}{2}\right) - 4Li(v) + Li(v^2)\Big] + \Big[\frac{11}{16}(3-v^2)(1+v^2) + \frac{v^4}{4}$$

$$- \frac{3}{2}v(3-v^2)\Big]$$

$$\times \ln\frac{1+v}{1-v} + 3v(3-v^2)\ln\frac{1+v}{2} - 2v(3-v^2)\ln v + \frac{3}{8}v(5-3v^2)\Big\}, \qquad (5)$$

where $Li(x)$ is the Euler dilogarithm (Refs. 2 and 3, for example), and we are using the known expression for the two-loop polarization of vacuum.[6,7] We calculate the corresponding contribution to the energy by integrating first over the exchange momentum and then over $\bar{v}$; taking the combinatorial factor of 2 into account (Fig. 1b), we find

$$\delta E_{2P} = \left(\frac{224}{15}\ln 2 - \frac{38}{15}\pi - \frac{118}{225}\right)\frac{\alpha^2(Z\alpha)}{\pi}E_F. \qquad (6)$$

We can calculate the contribution of the diagrams in Fig. 1c by means of the common expression which is derived in Ref. 3 for the radiative corrections to the two-photon emission from a fermion line. It is not difficult to see that in the case of an external field only the coefficient of the structure $\langle \gamma\hat{k}\gamma\rangle$ in the fermion factor[3] contributes to the hyperfine splitting. The approach of incorporating the insertions in an electron line corresponds to the following substitution in (1):

$$\frac{1}{k^2} \to \frac{\alpha}{2\pi}\int_0^1 dx \int_0^x dy \left[\frac{A_1(x,y)}{k^2y(1-y)+x^2} - \frac{k^2B_1(x,y)}{[k^2y(1-y)+x^2]^2}\right] \equiv \frac{\alpha}{2\pi}L(k), \qquad (7)$$

where

$$A_1(x,y) = (1-x)^2 - x - 2y\frac{1-x}{x} + \frac{2y^2}{x}\left(1-\frac{2}{x}\right),$$

$$B_1(x,y) = xy\left(1-\frac{x}{2}\right) + y^2\left(-\frac{4}{x}+1+x\right) + y^3\left(\frac{6}{x^2}-\frac{4}{x}-3\right) + 2\frac{y^4}{x}. \qquad (8)$$

Expressions (8) for A_1 amd B_1 differ slightly from those given in Ref. 3. Specifically, in the case of an external field with $k_0 = 0$ the decomposition of the electron factor into terms with the second and first powers of the denominator $k^2y(1-y) + x^2$ becomes ambiguous, since an integration by parts makes it possible to send one into the other. We have made use of this latitude to get rid of the logarithms in the expression for B_1 in Ref. 3 and thereby simplify the subsequent integration. Also taking into account the insertion of the vacuum polarization in the exchange photon and the combinatorial factor of 2, we find the following result for the contribution of the diagrams in Fig. 1c, associated with electron factor (7):

$$\delta E_{RP}^{(1)} = 8\,\frac{\alpha^2(Z\alpha)}{\pi^3}E_F\int_0^\infty dk\,k^2 L(k)\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}\;. \qquad (9)$$

After lengthy and tedious calculations we find

$$\delta E_{RP}^{(1)} = \left(-\frac{4}{3}\ln^2\frac{1+\sqrt{5}}{2} - \frac{20}{9}\sqrt{5}\ln\frac{1+\sqrt{5}}{2} - \frac{64}{45}\ln 2 + \frac{\pi^2}{9} + \frac{1043}{675}\right)\frac{\alpha^2(Z\alpha)}{\pi}E_F\,. \qquad (10)$$

The anomalous magnetic moment of the electron, which has been subtracted from expression (8) for the electron factor, must now be taken into account separately. It is easy to see that the corresponding contribution to the hyperfine splitting comes from the known correction of order $\alpha(Z\alpha)E_F$, which is induced by the polarization insertion in the exchange photon,[8,9] multipled by the anomalous moment:

$$\delta E_{RP}^{(2)} = \frac{3}{8}\,\frac{\alpha^2(Z\alpha)}{\pi}E_F\,. \qquad (11)$$

Combining (4), (6), (10), and (11), we find the resultant contribution of the diagrams in parts a–c of Fig. 1 to the hyperfine splitting:

$$\delta E = \left(-\frac{4}{3}\ln^2\frac{1+\sqrt{5}}{2} - \frac{20}{9}\sqrt{5}\ln\frac{1+\sqrt{5}}{2} + \frac{608}{45}\ln 2 + \frac{\pi^2}{9} - \frac{38}{15}\pi + \frac{91639}{37800}\right)\frac{\alpha^2(Z\alpha)}{\pi}E_F \approx 2.23\,\frac{\alpha^2(Z\alpha)}{\pi}E_F\,. \qquad (12)$$

As we mentioned earlier, correction (12) arises from an integration over virtual momenta which are small in comparison with the mass of a heavy particle if the structure of this heavy particle is not resolved. Result (12) thus applies not only to muonium but also to hydrogen, if the corresponding value is substituted in as the Fermi energy. Numerically, (12) is ~1.2 kHz for muonium and ~0.39 kHz for hydrogen. The effort to calculate the other corrections to the hyperfine splitting of order $\alpha^2(Z\alpha)E_F$, which do not contain polarization insertions in the external photons and which correspond to parts d–f in Fig. 1, is continuing. We hope to report its results soon.

All of the contributions to the hyperfine splitting derived in this study were found not only analytically but also through a numerical integration. The numerical calculations proved to be exceedingly useful for monitoring the validity of the analytic transformations. We are deeply indebted to V. G. Ivanov, G. A. Isaak'yan, and P. I. Okon for assistance in the numerical calculations.

[1]J. R. Sapirstein *et al.*, Phys. Rev. Lett. **51**, 982 (1983); Phys. Rev. D **29**, 2290 (1984).
[2]M. I. Eides *et al.*, Phys. Lett. B **177**, 425 (1986); S. G. Karshenboĭm *et al.*, Yad. Fiz. **44**, 1118 (1986)

[Sov. J. Nucl. Phys. **44**, 723 (1986)]; Zh. Eksp. Teor. Fiz. **65**, 664 (1987) [Sov. Phys. JETP **65**, 664 (1987)].
[3]M. I. Eides *et al.*, Phys. Lett. B **202**, 572 (1988); S. G. Karshenboĭm *et al.*, Zh. Eksp. Teor. Fiz. **94**, 42 (1988) [Sov. Phys. JETP **67**, 671 (1988)].
[4]F. G. Mariam *et al.*, Phys. Rev. Lett. **49**, 993 (1982).
[5]S. G. Karshenboĭm *et al.*, Yad. Fiz. **48**, 769 (1988) [Sov. J. Nucl. Phys. **48**, 490 (1988)].
[6]G. Källen and A. K. Sabry, Dan. Vid. Selsk. Mat.-Fys. Medd. **29**, No. 17 (1955).
[7]J. Schwinger, *Particles, Sources, and Fields*, Vol. 2, Mir, Moscow, 1976, p. 432.
[8]N. Kroll and F. Pollock, Phys. Rev. **84**, 595 (1951); Phys. Rev. **86**, 876 (1952).
[9]R. Karplus *et al.*, Phys. Rev. **84**, 597 (1951); R. Karplus and A. Klein, Phys. Rev. **85**, 972 (1952).

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